Proof. Let $f \in \mathrm{C}''(\mu;\nu)$. Then $f \circ \mu \circ f^{\dagger} \sqsubseteq \nu$; $f \circ \mu \circ f^{\dagger} \circ f \sqsubseteq \nu \circ f$; $f \circ \mu \circ 1_{\mathrm{Src}\, f} \sqsubseteq \nu \circ f$; $f \circ \mu \sqsubseteq \nu \circ f$; $f \in \mathrm{C}(\mu;\nu)$.

Let $f \in \mathrm{C}(\mu;\nu)$. Then $f \circ \mu \sqsubseteq \nu \circ f$; $f^{\dagger} \circ f \circ \mu \sqsubseteq f^{\dagger} \circ \nu \circ f$; $1_{\mathrm{Src}\, \mu} \circ \mu \sqsubseteq f^{\dagger} \circ \nu \circ f$; $\mu \sqsubseteq f^{\dagger} \circ \nu \circ f$; $f \in \mathrm{C}'(\mu;\nu)$. □

For entirely defined monovalued morphisms our three definitions of continuity coincide:

Theorem 10.5. If f is a monovalued and entirely defined morphism of a partially ordered dagger precategory then

$$f \in \mathrm{C}'(\mu;\nu) \Leftrightarrow f \in \mathrm{C}(\mu;\nu) \Leftrightarrow f \in \mathrm{C}''(\mu;\nu).$$

Proof. From two previous propositions. □

The classical general topology theorem that uniformly continuous function from a uniform space to an other uniform space is proximity-continuous regarding the proximities generated by the uniformities, generalized for reloids and funcoids takes the following form:

Theorem 10.6. If an entirely defined morphism of the category of reloids $f \in \mathrm{C}''(\mu;\nu)$ for some endomorphisms μ and ν of the category of reloids, then $(\mathsf{FCD}) f \in \mathrm{C}'((\mathsf{FCD})\mu; (\mathsf{FCD})\nu)$.

Exercise 10.1. I leave a simple exercise for the reader to prove the last theorem.

10.3 Continuity of a restricted morphism

Consider some partially ordered semigroup. (For example it can be the semigroup of funcoids or semigroup of reloids on some set regarding the composition.) Consider also some lattice (*lattice of objects*). (For example take the lattice of set theoretic filters.)

We will map every object A to so called *restricted identity* element I_A of the semigroup (for example restricted identity funcoid or restricted identity reloid). For identity elements we will require

1. $I_A \circ I_B = I_{A \sqcap B}$;
2. $f \circ I_A \sqsubseteq f$; $I_A \circ f \sqsubseteq f$.

In the case when our semigroup is "dagger" (that is is a dagger precategory) we will require also $(I_A)^{\dagger} = I_A$.

We can define restricting an element f of our semigroup to an object A by the formula $f|_A = f \circ I_A$.

We can define *rectangular restricting* an element f of our semigroup to objects A and B as $I_B \circ f \circ I_A$. Optionally we can define direct product $A \times B$ of two objects by the formula (true for funcoids and for reloids):

$$f \sqcap (A \times B) = I_B \circ f \circ I_A.$$

Square restricting of an element f to an object A is a special case of rectangular restricting and is defined by the formula $I_A \circ f \circ I_A$ (or by the formula $f \sqcap (A \times A)$).

Theorem 10.7. For every elements f, μ, ν our semigroup and an object A

1. $f \in \mathrm{C}(\mu;\nu) \Rightarrow f|_A \in \mathrm{C}(I_A \circ \mu \circ I_A;\nu)$;
2. $f \in \mathrm{C}'(\mu;\nu) \Rightarrow f|_A \in \mathrm{C}'(I_A \circ \mu \circ I_A;\nu)$;
3. $f \in \mathrm{C}''(\mu;\nu) \Rightarrow f|_A \in \mathrm{C}''(I_A \circ \mu \circ I_A;\nu)$.

(Two last items are true for the case when our semigroup is dagger.)

Proof.

1. $f|_A \in \mathrm{C}(I_A \circ \mu \circ I_A;\nu) \Leftrightarrow f|_A \circ I_A \circ \mu \circ I_A \sqsubseteq \nu \circ f|_A \Leftrightarrow f \circ I_A \circ I_A \circ \mu \circ I_A \sqsubseteq \nu \circ f|_A \Leftrightarrow f \circ I_A \circ \mu \circ I_A \sqsubseteq \nu \circ f \circ I_A \Leftarrow f \circ I_A \circ \mu \sqsubseteq \nu \circ f \Leftarrow f \circ \mu \sqsubseteq \nu \circ f \Leftrightarrow f \in \mathrm{C}(\mu;\nu)$.
2. $f|_A \in \mathrm{C}'(I_A \circ \mu \circ I_A;\nu) \Leftrightarrow I_A \circ \mu \circ I_A \sqsubseteq (f|_A)^{\dagger} \circ \nu \circ f|_A \Leftarrow I_A \circ \mu \circ I_A \sqsubseteq (f \circ I_A)^{\dagger} \circ \nu \circ f \circ I_A \Leftrightarrow I_A \circ \mu \circ I_A \sqsubseteq I_A \circ f^{\dagger} \circ \nu \circ f \circ I_A \Leftarrow \mu \sqsubseteq f^{\dagger} \circ \nu \circ f \Leftrightarrow f \in \mathrm{C}'(\mu;\nu)$.
3. $f|_A \in \mathrm{C}''(I_A \circ \mu \circ I_A;\nu) \Leftrightarrow f|_A \circ I_A \circ \mu \circ I_A \circ (f|_A)^{\dagger} \sqsubseteq \nu \Leftrightarrow f \circ I_A \circ I_A \circ \mu \circ I_A \circ I_A \circ f^{\dagger} \sqsubseteq \nu \Leftrightarrow f \circ I_A \circ \mu \circ I_A \circ f^{\dagger} \sqsubseteq \nu \Leftarrow f \circ \mu \circ f^{\dagger} \sqsubseteq \nu \Leftrightarrow f \in \mathrm{C}''(\mu;\nu)$. □