Chapter 10
Continuous morphisms

This chapter uses the apparatus from the section "Partially ordered dagger categories".

10.1 Traditional definitions of continuity

In this section we will show that having a funcoid or reloid $\uparrow f$ corresponding to a function f we can express continuity of it by the formula $\uparrow f \circ \mu \sqsubseteq \nu \circ \uparrow f$ (or similar formulas) where μ and ν are some spaces.

10.1.1 Pretopology

Let $(A; \mathrm{cl}_A)$ and $(B; \mathrm{cl}_B)$ be preclosure spaces. Then by definition a function $f: A \rightarrow B$ is continuous iff $f \,\mathrm{cl}_A(X) \subseteq \mathrm{cl}_B(f X)$ for every $X \in \mathscr{P} A$. Let now μ and ν be endofuncoids corresponding correspondingly to cl_A and cl_B. Then the condition for continuity can be rewritten as

$$\uparrow^{\mathsf{FCD}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu)} f \circ \mu \sqsubseteq \nu \circ \uparrow^{\mathsf{FCD}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu)} f.$$

10.1.2 Proximity spaces

Let μ and ν be proximity spaces (which I consider a special case of endofuncoids). By definition a function f is a proximity-continuous map (also called equicontinuous) from μ to ν iff

$$\forall X, Y \in \mathscr{P}(\mathrm{Ob}\,\mu): (X \,[\mu]^\ast\, Y \Rightarrow \langle f \rangle X \,[\nu]^\ast\, \langle f \rangle Y).$$

Equivalently transforming this formula we get (writing $\uparrow$ instead of $\uparrow^{\mathsf{FCD}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu)}$ for brevity):

$$\forall X, Y \in \mathscr{P}(\mathrm{Ob}\,\mu): \left(X \,[\mu]^\ast\, Y \Rightarrow \langle f \rangle Y \sqcap \langle \nu \rangle^\ast \langle f \rangle X \neq 0^{\mathfrak{F}(\mathrm{Dst}\,\nu)}\right);$$
$$\forall X, Y \in \mathscr{P}(\mathrm{Ob}\,\mu): \left(X \,[\mu]^\ast\, Y \Rightarrow \langle f \rangle Y \sqcap \langle \nu \circ \uparrow f \rangle^\ast X \neq 0^{\mathfrak{F}(\mathrm{Dst}\,\nu)}\right);$$
$$\forall X, Y \in \mathscr{P}(\mathrm{Ob}\,\mu): (X \,[\mu]^\ast\, Y \Rightarrow X \,[\nu \circ \uparrow f]^\ast\, \langle f \rangle Y);$$
$$\forall X, Y \in \mathscr{P}(\mathrm{Ob}\,\mu): (X \,[\mu]^\ast\, Y \Rightarrow \langle f \rangle Y \,[(\nu \circ \uparrow f)^{-1}]^\ast\, X);$$
$$\forall X, Y \in \mathscr{P}(\mathrm{Ob}\,\mu): (X \,[\mu]^\ast\, Y \Rightarrow \langle f \rangle Y \,[(\uparrow f)^{-1} \circ \nu^{-1}]^\ast\, X);$$
$$\forall X, Y \in \mathscr{P}(\mathrm{Ob}\,\mu): \left(X \,[\mu]^\ast\, Y \Rightarrow \uparrow^{\mathrm{Ob}\,\mu} X \sqcap \langle (\uparrow f)^{-1} \circ \nu^{-1} \rangle^\ast \langle f \rangle Y \neq 0^{\mathfrak{F}(\mathrm{Ob}\,\mu)}\right);$$
$$\forall X, Y \in \mathscr{P}(\mathrm{Ob}\,\mu): \left(X \,[\mu]^\ast\, Y \Rightarrow \uparrow^{\mathrm{Ob}\,\mu} X \sqcap \langle (\uparrow f)^{-1} \circ \nu^{-1} \circ \uparrow f \rangle^\ast Y \neq 0^{\mathfrak{F}(\mathrm{Ob}\,\mu)}\right);$$
$$\forall X, Y \in \mathscr{P}(\mathrm{Ob}\,\mu): (X \,[\mu]^\ast\, Y \Rightarrow Y \,[(\uparrow f)^{-1} \circ \nu^{-1} \circ \uparrow f]^\ast\, X);$$
$$\forall X, Y \in \mathscr{P}(\mathrm{Ob}\,\mu): (X \,[\mu]^\ast\, Y \Rightarrow X \,[(\uparrow f)^{-1} \circ \nu \circ \uparrow f]^\ast\, Y);$$
$$\mu \sqsubseteq (\uparrow f)^{-1} \circ \nu \circ \uparrow f.$$

So a function f is proximity continuous iff $\mu \sqsubseteq \left(\uparrow^{\mathsf{FCD}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu)} f\right)^{-1} \circ \nu \circ \uparrow^{\mathsf{FCD}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu)} f$.

10.1.3 Uniform spaces

Uniform spaces are a special case of endoreloids.

Let μ and ν be uniform spaces. By definition a function f is a uniformly continuous map from μ to ν iff

$$\forall \varepsilon \in \mathrm{GR}\,\nu \exists \delta \in \mathrm{GR}\,\nu \forall (x; y) \in \delta: (f x; f y) \in \varepsilon.$$