**Proof.** $g\circ(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B})\circ f=\bigsqcap\{\uparrow^{\mathsf{RLD}(\operatorname{Src}f;\operatorname{Dst}g)}(G\circ(A\times B)\circ F)\mid F\in\operatorname{GR}f,G\in\operatorname{GR}g,A\in\mathcal{A},B\in\mathcal{B}\}=\bigsqcap\{\uparrow^{\mathsf{RLD}(\operatorname{Src}f;\operatorname{Dst}g)}(\langle F^{-1}\rangle A\times\langle G\rangle B)\mid F\in\operatorname{GR}f,G\in\operatorname{GR}g,A\in\mathcal{A},B\in\mathcal{B}\}=\bigsqcap\{\uparrow^{\operatorname{Src}f}\langle F^{-1}\rangle A\times^{\mathsf{RLD}}\uparrow^{\operatorname{Dst}g}\langle G\rangle B\mid F\in\operatorname{GR}f,G\in\operatorname{GR}g,A\in\mathcal{A},B\in\mathcal{B}\}=$ (theorem 7.23) $=\bigsqcap\{\uparrow^{\operatorname{Src}f}\langle F^{-1}\rangle A\mid F\in\operatorname{GR}f,A\in\mathcal{A}\}\times^{\mathsf{RLD}}\bigsqcap\{\uparrow^{\operatorname{Dst}g}\langle G\rangle B\mid G\in\operatorname{GR}g,B\in\mathcal{B}\}=\bigsqcap\{\langle\uparrow^{\mathsf{FCD}(\operatorname{Dst}f;\operatorname{Src}f)}F^{-1}\rangle\uparrow^{\operatorname{Dst}f}A\mid F\in\operatorname{GR}f,A\in\mathcal{A}\}\times^{\mathsf{RLD}}\bigsqcap\{\langle\uparrow^{\mathsf{FCD}(\operatorname{Src}g;\operatorname{Dst}g)}G\rangle\uparrow^{\operatorname{Src}g}B\mid G\in\operatorname{GR}g,B\in\mathcal{B}\}=\bigsqcap\{\langle\uparrow^{\mathsf{FCD}(\operatorname{Dst}f;\operatorname{Src}f)}F^{-1}\rangle\mathcal{A}\mid F\in\operatorname{GR}f\}\times^{\mathsf{RLD}}\bigsqcap\{\langle\uparrow^{\mathsf{FCD}(\operatorname{Src}g;\operatorname{Dst}g)}G\rangle\mathcal{B}\mid G\in\operatorname{GR}g\}=$ (by definition of $(\mathsf{FCD})$) $=\langle(\mathsf{FCD})f^{-1}\rangle\mathcal{A}\times^{\mathsf{RLD}}\langle(\mathsf{FCD})g\rangle\mathcal{B}$. □

**Corollary 8.32.**

1. $(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B})\circ f=\langle(\mathsf{FCD})f^{-1}\rangle\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}$;
2. $g\circ(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B})=\mathcal{A}\times^{\mathsf{RLD}}\langle(\mathsf{FCD})g\rangle\mathcal{B}$.

## 8.3 Galois connections between funcoids and reloids

**Theorem 8.33.** $(\mathsf{FCD})\colon\mathsf{RLD}(A;B)\to\mathsf{FCD}(A;B)$ is the lower adjoint of $(\mathsf{RLD})_{\mathrm{in}}\colon\mathsf{FCD}(A;B)\to\mathsf{RLD}(A;B)$ for every sets $A$, $B$.

**Proof.** Because $(\mathsf{FCD})$ and $(\mathsf{RLD})_{\mathrm{in}}$ are trivially monotone, it's enough to prove (for every $f\in\mathsf{RLD}(A;B)$, $g\in\mathsf{FCD}(A;B)$)

$$f\sqsubseteq(\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD})f\quad\text{and}\quad(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}}g\sqsubseteq g.$$

The second formula follows from the fact that $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}}g=g$.

$$\begin{aligned}(\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD})f&=\\ \bigsqcup\{a\times^{\mathsf{RLD}}b\mid a\in\operatorname{atoms}^{\mathfrak{F}(A)},b\in\operatorname{atoms}^{\mathfrak{F}(B)},a\times^{\mathsf{FCD}}b\sqsubseteq(\mathsf{FCD})f\}&=\\ \bigsqcup\{a\times^{\mathsf{RLD}}b\mid a\in\operatorname{atoms}^{\mathfrak{F}(A)},b\in\operatorname{atoms}^{\mathfrak{F}(B)},a\,[(\mathsf{FCD})f]\,b\}&=\\ \bigsqcup\{a\times^{\mathsf{RLD}}b\mid a\in\operatorname{atoms}^{\mathfrak{F}(A)},b\in\operatorname{atoms}^{\mathfrak{F}(B)},a\times^{\mathsf{RLD}}b\not\asymp f\}&\sqsupseteq\\ \bigsqcup\{p\in\operatorname{atoms}(a\times^{\mathsf{RLD}}b)\mid a\in\operatorname{atoms}^{\mathfrak{F}(A)},b\in\operatorname{atoms}^{\mathfrak{F}(B)},p\not\asymp f\}&=\\ \bigsqcup\{p\in\operatorname{atoms}^{\mathsf{RLD}(A;B)}\mid p\not\asymp f\}&=\\ \bigsqcup\{p\mid p\in\operatorname{atoms}f\}=f.&\end{aligned}$$

□

**Corollary 8.34.**

1. $(\mathsf{FCD})\bigsqcup S=\bigsqcup\langle(\mathsf{FCD})\rangle S$ if $S\in\mathscr{P}\mathsf{RLD}(A;B)$.
2. $(\mathsf{RLD})_{\mathrm{in}}\bigsqcap S=\bigsqcap\langle(\mathsf{RLD})_{\mathrm{in}}\rangle S$ if $S\in\mathscr{P}\mathsf{FCD}(A;B)$.

**Proposition 8.35.** $(\mathsf{RLD})_{\mathrm{in}}(f\sqcap(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}))=((\mathsf{RLD})_{\mathrm{in}}f)\sqcap(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B})$ for every funcoid $f$ and $\mathcal{A}\in\mathfrak{F}(\operatorname{Src}f)$, $\mathcal{B}\in\mathfrak{F}(\operatorname{Dst}f)$.

**Proof.** $(\mathsf{RLD})_{\mathrm{in}}(f\sqcap(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}))=((\mathsf{RLD})_{\mathrm{in}}f)\sqcap(\mathsf{RLD})_{\mathrm{in}}(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})=((\mathsf{RLD})_{\mathrm{in}}f)\sqcap(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B})$. □

**Corollary 8.36.** $(\mathsf{RLD})_{\mathrm{in}}(f|_{\mathcal{A}})=((\mathsf{RLD})_{\mathrm{in}}f)|_{\mathcal{A}}$.

**Conjecture 8.37.** $(\mathsf{RLD})_{\mathrm{in}}$ is not a lower adjoint (in general).

**Conjecture 8.38.** $(\mathsf{RLD})_{\mathrm{out}}$ is neither a lower adjoint nor an upper adjoint (in general).

**Exercise 8.2.** Prove that $\operatorname{card}\mathsf{FCD}(A;B)=2^{2^{\max\{A,B\}}}$ if $A$ or $B$ is an infinite set (provided that $A$ and $B$ are nonempty).

**Lemma 8.39.** $\uparrow^{\mathsf{FCD}}\{(x;y)\}\sqsubseteq(\mathsf{FCD})g\Leftrightarrow\uparrow^{\mathsf{RLD}}\{(x;y)\}\sqsubseteq g$ for every reloid $g$.