Proposition 8.13. $\langle(\mathsf{FCD})f\rangle\mathcal{X}=\operatorname{im}(f|_{\mathcal{X}})$ for every reloid f and a filter $\mathcal{X}\in\mathfrak{F}(\operatorname{Src} f)$.

Proof. $\operatorname{im}(f|_{\mathcal{X}})=\operatorname{im}(\mathsf{FCD})(f|_{\mathcal{X}})=\operatorname{im}(((\mathsf{FCD})f)|_{\mathcal{X}})=\langle(\mathsf{FCD})f\rangle\mathcal{X}$. □

Proposition 8.14. $(\mathsf{FCD})f=\bigsqcup\left\{x\times^{\mathsf{FCD}}y\mid x\in\operatorname{atoms}^{\mathfrak{F}(\operatorname{Src} f)},y\in\operatorname{atoms}^{\mathfrak{F}(\operatorname{Src} f)},x\times^{\mathsf{RLD}}y\not\asymp f\right\}$ for every reloid f.

Proof. $(\mathsf{FCD})f=\bigsqcup\left\{x\times^{\mathsf{FCD}}y\mid x\in\operatorname{atoms}^{\mathfrak{F}(\operatorname{Src} f)},y\in\operatorname{atoms}^{\mathfrak{F}(\operatorname{Dst} f)},x\times^{\mathsf{FCD}}y\not\asymp(\mathsf{FCD})f\right\}$, but $x\times^{\mathsf{FCD}}y\not\asymp(\mathsf{FCD})f\Leftrightarrow x\,[(\mathsf{FCD})f]\,y\Leftrightarrow x\times^{\mathsf{RLD}}y\not\asymp f$, thus follows the theorem. □

8.2 Reloids induced by a funcoid

Every funcoid $f\in\mathsf{FCD}(A;B)$ induces a reloid from A to B in two ways, intersection of *outward* relations and union of *inward* reloidal products of filters:

$$(\mathsf{RLD})_{\text{out}}f=\bigsqcap\langle\uparrow^{\mathsf{RLD}}\rangle\operatorname{xyGR}f;$$
$$(\mathsf{RLD})_{\text{in}}f=\bigsqcup\{\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}\mid\mathcal{A}\in\mathfrak{F}(A),\mathcal{B}\in\mathfrak{F}(B),\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\sqsubseteq f\}.$$

Theorem 8.15. $(\mathsf{RLD})_{\text{in}}f=\bigsqcup\left\{a\times^{\mathsf{RLD}}b\mid a\in\operatorname{atoms}^{\mathfrak{F}(A)},b\in\operatorname{atoms}^{\mathfrak{F}(B)},a\times^{\mathsf{FCD}}b\sqsubseteq f\right\}$.

Proof. It follows from theorem 7.21. □

Remark 8.16. It seems that $(\mathsf{RLD})_{\text{in}}$ has smoother properties and is more important than $(\mathsf{RLD})_{\text{out}}$. (However see also the exercise below for $(\mathsf{RLD})_{\text{in}}$ not preserving identities.)

Proposition 8.17. $\operatorname{GR}\uparrow^{\mathsf{RLD}}f=\operatorname{GR}\uparrow^{\mathsf{FCD}}f$ for every Rel-morphism f.

Proof. $X\in\operatorname{GR}\uparrow^{\mathsf{RLD}}f\Leftrightarrow X\supseteq f\Leftrightarrow X\in\operatorname{GR}\uparrow^{\mathsf{FCD}}f$. □

Proposition 8.18. $(\mathsf{RLD})_{\text{out}}\uparrow^{\mathsf{FCD}}f=\uparrow^{\mathsf{RLD}}f$ for every Rel-morphism f.

Proof. $(\mathsf{RLD})_{\text{out}}\uparrow^{\mathsf{FCD}}f=\bigsqcap\langle\uparrow^{\mathsf{RLD}}\rangle\operatorname{xyGR}f=\uparrow^{\mathsf{RLD}}\min\operatorname{xyGR}f=\uparrow^{\mathsf{RLD}}f$ taking into account the previous proposition. □

Surprisingly, a funcoid is greater inward than outward:

Theorem 8.19. $(\mathsf{RLD})_{\text{out}}f\sqsubseteq(\mathsf{RLD})_{\text{in}}f$ for every funcoid f.

Proof. We need to prove

$$(\mathsf{RLD})_{\text{out}}f\sqsubseteq\bigsqcup\{\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}\mid\mathcal{A}\in\mathfrak{F}(\operatorname{Src} f),\mathcal{B}\in\mathfrak{F}(\operatorname{Dst} f),\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\sqsubseteq f\}.$$

Let

$$K\in\bigsqcup\{\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}\mid\mathcal{A}\in\mathfrak{F}(\operatorname{Src} f),\mathcal{B}\in\mathfrak{F}(\operatorname{Dst} f),\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\sqsubseteq f\}.$$

Then

$$\begin{aligned}K&\in\uparrow^{\mathsf{RLD}(\operatorname{Src} f;\operatorname{Dst} f)}\bigcup\{X_{\mathcal{A}}\times Y_{\mathcal{B}}\mid\mathcal{A}\in\mathfrak{F}(\operatorname{Src} f),\mathcal{B}\in\mathfrak{F}(\operatorname{Dst} f),\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\sqsubseteq f\}\\&=(\mathsf{RLD})_{\text{out}}\uparrow^{\mathsf{FCD}}\bigcup\{X_{\mathcal{A}}\times Y_{\mathcal{B}}\mid\mathcal{A}\in\mathfrak{F}(\operatorname{Src} f),\mathcal{B}\in\mathfrak{F}(\operatorname{Dst} f),\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\sqsubseteq f\}\\&=(\mathsf{RLD})_{\text{out}}\bigsqcup\{\uparrow^{\mathsf{FCD}}(X_{\mathcal{A}}\times Y_{\mathcal{B}})\mid\mathcal{A}\in\mathfrak{F}(\operatorname{Src} f),\mathcal{B}\in\mathfrak{F}(\operatorname{Dst} f),\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\sqsubseteq f\}\\&\sqsupseteq(\mathsf{RLD})_{\text{out}}\bigsqcup\operatorname{atoms}f\\&=(\mathsf{RLD})_{\text{out}}f\end{aligned}$$

where $X_{\mathcal{A}}\in\mathcal{A}$, $Y_{\mathcal{B}}\in\mathcal{B}$. So $K\in(\mathsf{RLD})_{\text{out}}f$. □

Theorem 8.20. $(\mathsf{FCD})(\mathsf{RLD})_{\text{in}}f=f$ for every funcoid f.