Theorem 8.3. $(\mathsf{FCD})f = \bigsqcap \langle \uparrow^{\mathsf{FCD}} \rangle \mathrm{xyGR}\, f$ for every reloid f.

Proof. Let a be an ultrafilter on $\mathrm{Src}\, f$.

$\langle (\mathsf{FCD}) f\rangle a = \bigsqcap \{ \langle \uparrow^{\mathsf{FCD}} F\rangle a \mid F \in \mathrm{xyGR}\, f\}$ by the definition of (FCD).

$\langle \bigsqcap \langle \uparrow^{\mathsf{FCD}} \rangle \mathrm{xyGR}\, f\rangle a = \bigsqcap \{ \langle \uparrow^{\mathsf{FCD}} F\rangle a \mid F \in \mathrm{xyGR}\, f\}$ by theorem 6.68.

So $\langle (\mathsf{FCD}) f\rangle a = \langle \bigsqcap \langle \uparrow^{\mathsf{FCD}} \rangle \mathrm{xyGR}\, f\rangle a$ for every ultrafilter a. □

Lemma 8.4. For every two filter bases S and T of morphisms $\mathrm{Rel}(U;V)$ and every set $A \subseteq U$

$$\bigsqcap \langle \uparrow^{\mathsf{RLD}} \rangle S = \bigsqcap \langle \uparrow^{\mathsf{RLD}} \rangle T \Rightarrow \bigsqcap \{ \uparrow^V \langle F\rangle A \mid F \in S\} = \bigsqcap \{ \uparrow^V \langle G\rangle A \mid G \in T\}.$$

Proof. Let $\bigsqcap \langle \uparrow^{\mathsf{RLD}} \rangle S = \bigsqcap \langle \uparrow^{\mathsf{RLD}} \rangle T$.

First let prove that $\{ \langle F\rangle A \mid F \in S\}$ is a filter base. Let $X, Y \in \{ \langle F\rangle A \mid F \in S\}$. Then $X = \langle F_X\rangle A$ and $Y = \langle F_Y\rangle A$ for some $F_X, F_Y \in S$. Because S is a filter base, we have $S \ni F_Z \sqsubseteq F_X \sqcap F_Y$. So $\langle F_Z\rangle A \sqsubseteq X \sqcap Y$ and $\langle F_Z\rangle A \in \{ \langle F\rangle A \mid F \in S\}$. So $\{ \langle F\rangle A \mid F \in S\}$ is a filter base.

Suppose $X \in \bigsqcap \{ \uparrow^V \langle F\rangle A \mid F \in S\}$. Then there exists $X' \in \{ \langle F\rangle A \mid F \in S\}$ where $X \sqsupseteq X'$ because $\{ \langle F\rangle A \mid F \in S\}$ is a filter base. That is $X' = \langle F\rangle A$ for some $F \in S$. There exists $G \in T$ such that $G \sqsubseteq F$ because T is a filter base. Let $Y' = \langle G\rangle A$. We have $Y' \sqsubseteq X' \sqsubseteq X$; $Y' \in \{ \langle G\rangle A \mid G \in T\}$; $Y' \in \bigsqcap \{ \uparrow^V \langle G\rangle A \mid G \in T\}$; $X \in \bigsqcap \{ \uparrow^V \langle G\rangle A \mid G \in T\}$. The reverse is symmetric. □

Lemma 8.5. $\{ G \circ F \mid F \in \mathrm{GR}\, f, G \in \mathrm{GR}\, g\}$ is a filter base for every reloids f and g.

Proof. Let denote $D = \{ G \circ F \mid F \in \mathrm{GR}\, f, G \in \mathrm{GR}\, g\}$. Let $A \in D \wedge B \in D$. Then $A = G_A \circ F_A \wedge B = G_B \circ F_B$ for some $F_A, F_B \in \mathrm{GR}\, f$, $G_A, G_B \in \mathrm{GR}\, g$. So $A \cap B \supseteq (G_A \cap G_B) \circ (F_A \cap F_B) \in D$ because $F_A \cap F_B \in \mathrm{GR}\, f$ and $G_A \cap G_B \in \mathrm{GR}\, g$. □

Theorem 8.6. $(\mathsf{FCD})(g \circ f) = ((\mathsf{FCD})g) \circ ((\mathsf{FCD})f)$ for every composable reloids f and g.

Proof.

$$\begin{aligned}\langle (\mathsf{FCD})(g \circ f)\rangle^\ast X &= \bigsqcap \{ \uparrow^{\mathrm{Dst}\, g} \langle H\rangle X \mid H \in \mathrm{GR}(g \circ f)\} \\ &= \bigsqcap \left\{ \uparrow^{\mathrm{Dst}\, g} \langle H\rangle X \mid H \in \mathrm{GR} \bigsqcap \{ \uparrow^{\mathsf{RLD}} (G \circ F) \mid F \in \mathrm{xyGR}\, f, G \in \mathrm{xyGR}\, g\} \right\}.\end{aligned}$$

Obviously

$$\bigsqcap \{ \uparrow^{\mathsf{RLD}} (G \circ F) \mid F \in \mathrm{xyGR}\, f, G \in \mathrm{xyGR}\, g\} = \bigsqcap \langle \uparrow^{\mathsf{RLD}} \rangle \mathrm{xyGR} \bigsqcap \{ \uparrow^{\mathsf{RLD}} (G \circ F) \mid F \in \mathrm{xyGR}\, f, G \in \mathrm{xyGR}\, g\};$$

from this by lemma 8.4 (taking into account that

$$\{ G \circ F \mid F \in \mathrm{GR}\, f, G \in \mathrm{GR}\, g\}$$

and

$$\mathrm{GR} \bigsqcap \{ \uparrow^{\mathsf{RLD}} (G \circ F) \mid F \in \mathrm{xyGR}\, f, G \in \mathrm{xyGR}\, g\}$$

are filter bases)

$$\bigsqcap \left\{ \uparrow^{\mathrm{Dst}\, g} \langle H\rangle X \mid H \in \mathrm{GR} \bigsqcap \{ \uparrow^{\mathsf{RLD}} (G \circ F) \mid F \in \mathrm{xyGR}\, f, G \in \mathrm{xyGR}\, g\} \right\} = \bigsqcap \{ \uparrow^{\mathrm{Dst}\, g} \langle G \circ F\rangle X \mid F \in \mathrm{GR}\, f, G \in \mathrm{GR}\, g\}.$$

On the other side

$$\begin{aligned}\langle ((\mathsf{FCD})g) \circ ((\mathsf{FCD})f)\rangle^\ast X &= \langle (\mathsf{FCD})g\rangle \langle (\mathsf{FCD})f\rangle^\ast X \\ &= \langle (\mathsf{FCD})g\rangle \bigsqcap \{ \uparrow^{\mathrm{Dst}\, g} \langle F\rangle X \mid F \in \mathrm{xyGR}\, f\} \\ &= \bigsqcap \left\{ \langle \uparrow^{\mathsf{FCD}} G\rangle \bigsqcap \{ \uparrow^{\mathrm{Dst}\, g} \langle F\rangle X \mid F \in \mathrm{xyGR}\, f\} \mid G \in \mathrm{xyGR}\, g\right\}.\end{aligned}$$

Let's prove that $\{ \langle F\rangle X \mid F \in \mathrm{xyGR}\, f\}$ is a filter base. If $A, B \in \{ \langle F\rangle X \mid F \in \mathrm{xyGR}\, f\}$ then $A = \langle F_1\rangle X$, $B = \langle F_2\rangle X$ where $F_1, F_2 \in \mathrm{xyGR}\, f$. $A \cap B \supseteq \langle F_1 \sqcap F_2\rangle X \in \{ \langle F\rangle X \mid F \in \mathrm{xyGR}\, f\}$. So $\{ \langle F\rangle X \mid F \in \mathrm{xyGR}\, f\}$ is really a filter base.