Theorem 7.56. Atoms of the lattice $\mathrm{Compl}\mathsf{RLD}(A;B)$ are exactly reloidal products of the form $\uparrow^A\{\alpha\}\times^{\mathsf{RLD}}b$ where $\alpha\in A$ and b is an ultrafilter on B.

Proof. First, it's easy to see that $\uparrow^A\{\alpha\}\times^{\mathsf{RLD}}b$ are elements of $\mathrm{Compl}\mathsf{RLD}(A;B)$. Also $0^{\mathsf{RLD}(A;B)}$ is an element of $\mathrm{Compl}\mathsf{RLD}(A;B)$.

$\uparrow^A\{\alpha\}\times^{\mathsf{RLD}}b$ are atoms of $\mathrm{Compl}\mathsf{RLD}(A;B)$ because they are atoms of $\mathsf{RLD}(A;B)$.

It remains to prove that if f is an atom of $\mathrm{Compl}\mathsf{RLD}(A;B)$ then $f=\uparrow^A\{\alpha\}\times^{\mathsf{RLD}}b$ for some $\alpha\in A$ and an ultrafilter b on B.

Suppose f is a non-empty complete reloid. Then $\uparrow^A\{\alpha\}\times^{\mathsf{RLD}}b\sqsubseteq f$ for some $\alpha\in A$ and an ultrafilter b on B. If f is an atom then $f=\uparrow^A\{\alpha\}\times^{\mathsf{RLD}}b$. □

Obvious 7.57. $\mathrm{Compl}\mathsf{RLD}(A;B)$ is an atomistic lattice.

Proposition 7.58. $\mathrm{Compl}\,f=\bigsqcup\left\{f|_{\uparrow^{\mathrm{Src}\,f}\{\alpha\}}\mid\alpha\in\mathrm{Src}\,f\right\}$ for every reloid f.

Proof. Let's denote R the right part of the equality to be proven.

That R is a complete reloid follows from the equality

$$f|_{\uparrow^{\mathrm{Src}\,f}\{\alpha\}}=\uparrow^{\mathrm{Src}\,f}\{\alpha\}\times^{\mathsf{RLD}}\mathrm{im}\left(f|_{\uparrow^{\mathrm{Src}\,f}\{\alpha\}}\right).$$

The only thing left to prove is that $g\sqsubseteq R$ for every complete reloid g such that $g\sqsubseteq f$.

Really let g be a complete reloid such that $g\sqsubseteq f$. Then

$$g=\bigsqcup\left\{\uparrow^{\mathrm{Src}\,f}\{\alpha\}\times^{\mathsf{RLD}}G(\alpha)\mid\alpha\in\mathrm{Src}\,f\right\}$$

for some function $G:\mathrm{Src}\,f\to\mathfrak{F}(\mathrm{Dst}\,f)$.

We have $\uparrow^{\mathrm{Src}\,f}\{\alpha\}\times^{\mathsf{RLD}}G(\alpha)=g|_{\uparrow^{\mathrm{Src}\,f}\{\alpha\}}\sqsubseteq f|_{\uparrow^{\mathrm{Src}\,f}\{\alpha\}}$. Thus $g\sqsubseteq R$. □

Conjecture 7.59. $\mathrm{Compl}\,f\sqcap\mathrm{Compl}\,g=\mathrm{Compl}(f\sqcap g)$ for every $f,g\in\mathsf{RLD}(A;B)$.

Theorem 7.60. $\mathrm{Compl}\bigsqcup R=\bigsqcup\langle\mathrm{Compl}\rangle R$ for every set $R\in\mathscr{P}\mathsf{RLD}(A;B)$ for every sets A, B.

Proof.

$$\begin{aligned}\mathrm{Compl}\bigsqcup R &=\\ \bigsqcup\left\{\left(\bigsqcup R\right)|_{\uparrow^A\{\alpha\}}\mid\alpha\in A\right\} &= \quad\text{(proposition 4.194)}\\ \bigsqcup\left\{\bigsqcup\left\{f|_{\uparrow^A\{\alpha\}}\mid\alpha\in A\right\}\mid f\in R\right\} &=\\ \bigsqcup\langle\mathrm{Compl}\rangle R.\end{aligned}$$

□

Lemma 7.61. Completion of a co-complete reloid is principal.

Proof. Let f be a co-complete reloid. Then there is a function $F:\mathrm{Dst}\,f\to\mathfrak{F}(\mathrm{Src}\,f)$ such that

$$f=\bigsqcup\left\{F(\alpha)\times^{\mathsf{RLD}}\uparrow^{\mathrm{Dst}\,f}\{\alpha\}\mid\alpha\in\mathrm{Dst}\,f\right\}.$$

So

$$\begin{aligned}\mathrm{Compl}\,f &=\\ \bigsqcup\left\{\left(\bigsqcup\left\{F(\alpha)\times^{\mathsf{RLD}}\uparrow^{\mathrm{Dst}\,f}\{\alpha\}\mid\alpha\in\mathrm{Dst}\,f\right\}\right)|_{\uparrow^{\mathrm{Src}\,f}\{\beta\}}\mid\beta\in\mathrm{Src}\,f\right\} &=\\ \bigsqcup\left\{\left(\bigsqcup\left\{F(\alpha)\times^{\mathsf{RLD}}\uparrow^{\mathrm{Dst}\,f}\{\alpha\}\mid\alpha\in\mathrm{Dst}\,f\right\}\right)\sqcap\left(\uparrow^{\mathrm{Src}\,f}\{\beta\}\times^{\mathsf{RLD}}1^{\mathfrak{F}(\mathrm{Dst}\,f)}\right)\mid\beta\in\mathrm{Src}\,f\right\} &= \quad (*)\\ \bigsqcup\left\{\bigsqcup\left\{\left(F(\alpha)\times^{\mathsf{RLD}}\uparrow^{\mathrm{Dst}\,f}\{\alpha\}\right)\sqcap\left(\uparrow^{\mathrm{Src}\,f}\{\beta\}\times^{\mathsf{RLD}}1^{\mathfrak{F}(\mathrm{Dst}\,f)}\right)\mid\alpha\in\mathrm{Dst}\,f\right\}\mid\beta\in\mathrm{Src}\,f\right\} &=\\ \bigsqcup\left\{\bigsqcup\left\{\uparrow^{\mathrm{Src}\,f}\{\beta\}\times^{\mathsf{RLD}}\uparrow^{\mathrm{Dst}\,f}\{\alpha\}\mid\alpha\in\mathrm{Dst}\,f\right\}\mid\beta\in\mathrm{Src}\,f,\uparrow^{\mathrm{Src}\,f}\{\beta\}\sqsubseteq F(\alpha)\right\}.\end{aligned}$$

* proposition 4.194.

Thus $\mathrm{Compl}\,f$ is principal. □

Theorem 7.62. $\mathrm{Compl}\,\mathrm{CoCompl}\,f=\mathrm{CoCompl}\,\mathrm{Compl}\,f=\mathrm{Cor}\,f$ for every reloid f.