Obvious 7.49. Principal reloids are complete and co-complete.

Obvious 7.50. Join (on the lattice of reloids) of complete reloids is complete.

Corollary 7.51. ComplRLD (with the induced order) is a complete lattice.

Theorem 7.52. A reloid which is both complete and co-complete is principal.

Proof. Let f be a complete and co-complete reloid. We have

$$f = \bigsqcup \{\uparrow^{\operatorname{Src} f}\{\alpha\} \times^{\mathsf{RLD}} G(\alpha) \mid \alpha \in \operatorname{Src} f\} \quad \text{and} \quad f = \bigsqcup \{H(\beta) \times^{\mathsf{RLD}} \uparrow^{\operatorname{Dst} f}\{\beta\} \mid \beta \in \operatorname{Dst} f\}$$

for some functions $G\colon \operatorname{Src} f \to \mathfrak{F}(\operatorname{Dst} f)$ and $H\colon \operatorname{Dst} f \to \mathfrak{F}(\operatorname{Src} f)$. For every $\alpha \in \operatorname{Src} f$ we have

$$\begin{aligned}
G(\alpha) &= \\
\operatorname{im} f|_{\uparrow^{\operatorname{Src} f}\{\alpha\}} &= \\
\operatorname{im}\left(f \sqcap \left(\uparrow^{\operatorname{Src} f}\{\alpha\} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\operatorname{Dst} f)}\right)\right) &= (*)\\
\operatorname{im} \bigsqcup \left\{(H(\beta) \times^{\mathsf{RLD}} \uparrow^{\operatorname{Dst} f}\{\beta\}) \sqcap \left(\uparrow^{\operatorname{Src} f}\{\alpha\} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\operatorname{Dst} f)}\right) \mid \beta \in \operatorname{Dst} f\right\} &= \\
\operatorname{im} \bigsqcup \{(H(\beta) \sqcap \uparrow^{\operatorname{Src} f}\{\alpha\}) \times^{\mathsf{RLD}} \uparrow^{\operatorname{Dst} f}\{\beta\} \mid \beta \in \operatorname{Dst} f\} &= \\
\operatorname{im} \bigsqcup \left\{\left(\begin{cases} \uparrow^{\operatorname{Src} f}\{\alpha\} \times^{\mathsf{RLD}} \uparrow^{\operatorname{Dst} f}\{\beta\} & \text{if } H(\beta) \not\asymp \uparrow^{\operatorname{Src} f}\{\alpha\} \\ 0^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)} & \text{if } H(\beta) \asymp \uparrow^{\operatorname{Src} f}\{\alpha\} \end{cases}\right) \mid \beta \in \operatorname{Dst} f\right\} &= \\
\operatorname{im} \bigsqcup \{\uparrow^{\operatorname{Src} f}\{\alpha\} \times^{\mathsf{RLD}} \uparrow^{\operatorname{Dst} f}\{\beta\} \mid \beta \in \operatorname{Dst} f, H(\beta) \not\asymp \uparrow^{\operatorname{Src} f}\{\alpha\}\} &= \\
\operatorname{im} \bigsqcup \{\uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)}\{(\alpha; \beta)\} \mid \beta \in \operatorname{Dst} f, H(\beta) \not\asymp \uparrow^{\operatorname{Src} f}\{\alpha\}\} &= \\
\bigsqcup \{\uparrow^{\operatorname{Dst} f}\{\beta\} \mid \beta \in \operatorname{Dst} f, H(\beta) \not\asymp \uparrow^{\operatorname{Src} f}\{\alpha\}\}
\end{aligned}$$

* proposition 4.194 was used.

Thus $G(\alpha)$ is a principal filter that is $G(\alpha) = \uparrow^{\operatorname{Dst} f} g(\alpha)$ for some $g\colon \operatorname{Src} f \to \operatorname{Dst} f$; $\uparrow^{\operatorname{Src} f}\{\alpha\} \times^{\mathsf{RLD}} G(\alpha) = \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)}(\{\alpha\} \times g(\alpha))$; f is principal as a join of principal reloids. □

Conjecture 7.53. Composition of complete reloids is complete. [TODO: Solved.]

Theorem 7.54.

1. For a complete reloid f there exists exactly one function $F \in \mathfrak{F}(\operatorname{Dst} f)^{\operatorname{Src} f}$ such that
$$f = \bigsqcup \{\uparrow^{\operatorname{Src} f}\{\alpha\} \times^{\mathsf{RLD}} F(\alpha) \mid \alpha \in \operatorname{Src} f\}.$$
2. For a co-complete reloid f there exists exactly one function $F \in \mathfrak{F}(\operatorname{Src} f)^{\operatorname{Dst} f}$ such that
$$f = \bigsqcup \{F(\alpha) \times^{\mathsf{RLD}} \uparrow^{\operatorname{Dst} f}\{\alpha\} \mid \alpha \in \operatorname{Dst} f\}.$$

Proof. We will prove only the first as the second is similar. Let

$$f = \bigsqcup \{\uparrow^{\operatorname{Src} f}\{\alpha\} \times^{\mathsf{RLD}} F(\alpha) \mid \alpha \in \operatorname{Src} f\} = \bigsqcup \{\uparrow^{\operatorname{Src} f}\{\alpha\} \times^{\mathsf{RLD}} G(\alpha) \mid \alpha \in \operatorname{Src} f\}$$

for some $F, G \in \mathfrak{F}(\operatorname{Dst} f)^{\operatorname{Src} f}$. We need to prove $F = G$. Let $\beta \in \operatorname{Src} f$.

$$\begin{aligned}
f \sqcap \left(\uparrow^{\operatorname{Src} f}\{\beta\} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\operatorname{Dst} f)}\right) &= \text{(proposition 4.194)}\\
\bigsqcup \left\{(\uparrow^{\operatorname{Src} f}\{\alpha\} \times^{\mathsf{RLD}} F(\alpha)) \sqcap \left(\uparrow^{\operatorname{Src} f}\{\beta\} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\operatorname{Dst} f)}\right) \mid \alpha \in \operatorname{Src} f\right\} &= \\
\uparrow^{\operatorname{Src} f}\{\beta\} \times^{\mathsf{RLD}} F(\beta).
\end{aligned}$$

Similarly $f \sqcap \left(\uparrow^{\operatorname{Src} f}\{\beta\} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\operatorname{Dst} f)}\right) = \uparrow^{\operatorname{Src} f}\{\beta\} \times^{\mathsf{RLD}} G(\beta)$. Thus $\uparrow^{\operatorname{Src} f}\{\beta\} \times^{\mathsf{RLD}} F(\beta) = \uparrow^{\operatorname{Src} f}\{\beta\} \times^{\mathsf{RLD}} G(\beta)$ and so $F(\beta) = G(\beta)$. □

Definition 7.55. *Completion* and *co-completion* of a reloid $f \in \mathsf{RLD}(A; B)$ are defined by the formulas:

$$\operatorname{Compl} f = \operatorname{Cor}^{(\mathsf{RLD}(A;B); \operatorname{Compl}\mathsf{RLD}(A;B))} f; \quad \operatorname{CoCompl} f = \operatorname{Cor}^{(\mathsf{RLD}(A;B); \operatorname{CoCompl}\mathsf{RLD}(A;B))} f.$$