Theorem 7.32. $\mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}} = \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))} \mathrm{id}_A \mid A \in \mathcal{A} \right\}$ for every filter $\mathcal{A}$.

Proof. Let $K \in \mathrm{GR} \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))} \mathrm{id}_A \mid A \in \mathcal{A} \right\}$, then there exists $A \in \mathcal{A}$ such that $K \supseteq \mathrm{id}_A$. Then

$\mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}} \sqsubseteq \uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))} \mathrm{id}_{\mathrm{Base}(\mathcal{A})} \sqcap \left(\mathcal{A} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\mathrm{Base}(\mathcal{A}))} \right) \sqsubseteq \uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))} \mathrm{id}_{\mathrm{Base}(\mathcal{A})} \sqcap \left(\uparrow^{\mathrm{Base}(\mathcal{A})} A \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\mathrm{Base}(\mathcal{A}))} \right) = \uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))} \mathrm{id}_{\mathrm{Base}(\mathcal{A})} \sqcap \uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))} (A \times \mathrm{Base}(\mathcal{A})) = \uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))} (\mathrm{id}_{\mathrm{Base}(\mathcal{A})} \cap (A \times \mathrm{Base}(\mathcal{A}))) = \uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))} \mathrm{id}_A \sqsubseteq \uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))} K$.

Thus $K \in \mathrm{GR}\, \mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}}$.

Reversely let $K \in \mathrm{GR}\, \mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}} = \mathrm{GR}\left(\mathrm{id}^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A}))} \sqcap \left(\mathcal{A} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\mathrm{Base}(\mathcal{A}))} \right) \right)$, then there exists $A \in \mathcal{A}$ such that $K \in \mathrm{GR} \uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))} (\mathrm{id}_{\mathrm{Base}(\mathcal{A})} \cap (A \times \mathrm{Base}(\mathcal{A}))) = \mathrm{GR} \uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))} \mathrm{id}_A \sqsupseteq \mathrm{GR} \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))} \mathrm{id}_A \mid A \in \mathcal{A} \right\}$. □

Corollary 7.33. $(\mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}})^{-1} = \mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}}$.

Theorem 7.34. $f|_{\mathcal{A}} = f \circ \mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}}$ for every reloid f and $\mathcal{A} \in \mathfrak{F}(\mathrm{Src}\, f)$.

Proof. We need to prove that $f \sqcap \left(\mathcal{A} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\mathrm{Dst}\, f)} \right) = f \circ \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(\mathrm{Src}\, f;\mathrm{Src}\, f)} \mathrm{id}_A \mid A \in \mathcal{A} \right\}$.

We have $f \circ \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(\mathrm{Src}\, f;\mathrm{Src}\, f)} \mathrm{id}_A \mid A \in \mathcal{A} \right\} = \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(\mathrm{Src}\, f;\mathrm{Dst}\, f)} (F \circ \mathrm{id}_A) \mid F \in \mathrm{GR}\, f, A \in \mathcal{A} \right\} = \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(\mathrm{Src}\, f;\mathrm{Dst}\, f)} (F|_A) \mid F \in \mathrm{GR}\, f, A \in \mathcal{A} \right\} = \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(\mathrm{Src}\, f;\mathrm{Dst}\, f)} (F \cap (A \times \mathrm{Dst}\, f)) \mid F \in \mathrm{GR}\, f, A \in \mathcal{A} \right\} = \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(\mathrm{Src}\, f;\mathrm{Dst}\, f)} F \mid F \in \mathrm{GR}\, f \right\} \sqcap \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(\mathrm{Src}\, f;\mathrm{Dst}\, f)} (A \times \mathrm{Dst}\, f) \mid A \in \mathcal{A} \right\} = f \sqcap \left(\mathcal{A} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\mathrm{Dst}\, f)} \right)$. □

Theorem 7.35. $(g \circ f)|_{\mathcal{A}} = g \circ (f|_{\mathcal{A}})$ for every composable reloids f and g and $\mathcal{A} \in \mathfrak{F}(\mathrm{Src}\, f)$.

Proof. $(g \circ f)|_{\mathcal{A}} = (g \circ f) \circ \mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}} = g \circ (f \circ \mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}}) = g \circ (f|_{\mathcal{A}})$. □

Theorem 7.36. $f \sqcap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = \mathrm{id}^{\mathsf{RLD}}_{\mathcal{B}} \circ f \circ \mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}}$ for every reloid f and $\mathcal{A} \in \mathfrak{F}(\mathrm{Src}\, f)$, $\mathcal{B} \in \mathfrak{F}(\mathrm{Dst}\, f)$.

Proof. $f \sqcap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = f \sqcap \left(\mathcal{A} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\mathrm{Dst}\, f)} \right) \sqcap \left(1^{\mathfrak{F}(\mathrm{Src}\, f)} \times^{\mathsf{RLD}} \mathcal{B} \right) = f|_{\mathcal{A}} \sqcap \left(1^{\mathfrak{F}(\mathrm{Src}\, f)} \times^{\mathsf{RLD}} \mathcal{B} \right) = (f \circ \mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}}) \sqcap \left(1^{\mathfrak{F}(\mathrm{Src}\, f)} \times^{\mathsf{RLD}} \mathcal{B} \right) = \left((f \circ \mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}})^{-1} \sqcap \left(1^{\mathfrak{F}(\mathrm{Src}\, f)} \times^{\mathsf{RLD}} \mathcal{B} \right)^{-1} \right)^{-1} = \left((\mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}} \circ f^{-1}) \sqcap \left(\mathcal{B} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\mathrm{Src}\, f)} \right) \right)^{-1} = (\mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}} \circ f^{-1} \circ \mathrm{id}^{\mathsf{RLD}}_{\mathcal{B}})^{-1} = \mathrm{id}^{\mathsf{RLD}}_{\mathcal{B}} \circ f \circ \mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}}$. □

Theorem 7.37. $f|_{\uparrow^{\mathrm{Src}}\{\alpha\}} = \uparrow^{\mathrm{Src}}\{\alpha\} \times^{\mathsf{RLD}} \mathrm{im}(f|_{\uparrow^{\mathrm{Src}}\{\alpha\}})$ for every reloid f and $\alpha \in \mathrm{Src}\, f$.

Proof. First,

$$
\begin{aligned}
\mathrm{im}(f|_{\uparrow^{\mathrm{Src}}\{\alpha\}}) &= \\
\bigsqcap \langle \uparrow^{\mathrm{Dst}\, f} \rangle \langle \mathrm{im} \rangle \mathrm{GR}(f|_{\uparrow^{\mathrm{Src}}\{\alpha\}}) &= \\
\bigsqcap \langle \uparrow^{\mathrm{Dst}\, f} \rangle \langle \mathrm{im} \rangle \mathrm{GR}\left(f \sqcap \left(\uparrow^{\mathrm{Src}\, f} \{\alpha\} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\mathrm{Dst}\, f)} \right) \right) &= \\
\bigsqcap \left\{ \uparrow^{\mathrm{Dst}\, f} \mathrm{im}(F \cap (\{\alpha\} \times \mathrm{Dst}\, f)) \mid F \in \mathrm{GR}\, f \right\} &= \\
\bigsqcap \left\{ \uparrow^{\mathrm{Dst}\, f} \mathrm{im}(F|_{\{\alpha\}}) \mid F \in \mathrm{GR}\, f \right\}&.
\end{aligned}
$$

Taking this into account we have:

$$
\begin{aligned}
\uparrow^{\mathrm{Src}}\{\alpha\} \times^{\mathsf{RLD}} \mathrm{im}(f|_{\uparrow^{\mathrm{Src}}\{\alpha\}}) &= \\
\bigsqcap \left\{ \uparrow^{\mathsf{RLD}(\mathrm{Src}\, f;\mathrm{Dst}\, f)} (\{\alpha\} \times K) \mid K \in \mathrm{im}(f|_{\uparrow^{\mathrm{Src}}\{\alpha\}}) \right\} &= \\
\bigsqcap \left\{ \uparrow^{\mathsf{RLD}(\mathrm{Src}\, f;\mathrm{Dst}\, f)} (\{\alpha\} \times \mathrm{im}(F|_{\{\alpha\}})) \mid F \in \mathrm{GR}\, f \right\} &= \\
\bigsqcap \left\{ \uparrow^{\mathsf{RLD}(\mathrm{Src}\, f;\mathrm{Dst}\, f)} (F|_{\{\alpha\}}) \mid F \in \mathrm{GR}\, f \right\} &= \\
\bigsqcap \left\{ \uparrow^{\mathsf{RLD}(\mathrm{Src}\, f;\mathrm{Dst}\, f)} (F \cap (\{\alpha\} \times \mathrm{Dst}\, f)) \mid F \in \mathrm{GR}\, f \right\} &= \\
\bigsqcap \left\{ \uparrow^{\mathsf{RLD}(\mathrm{Src}\, f;\mathrm{Dst}\, f)} F \mid F \in \mathrm{GR}\, f \right\} \sqcap \uparrow^{\mathsf{RLD}(\mathrm{Src}\, f;\mathrm{Dst}\, f)} (\{\alpha\} \times \mathrm{Dst}\, f) &= \\
f \sqcap \uparrow^{\mathsf{RLD}(\mathrm{Src}\, f;\mathrm{Dst}\, f)} (\{\alpha\} \times \mathrm{Dst}\, f) &= \\
f|_{\uparrow^{\mathrm{Src}}\{\alpha\}}&.
\end{aligned}
$$

□