From this follows (2).

(6)⇒(5). $\langle f\rangle^*\bigcup S = \bigsqcup\{\langle f\rangle^*\{a\}\mid a\in\bigcup S\} = \bigsqcup\bigcup\{\{\langle f\rangle^*\{a\}\mid a\in A\}\mid A\in S\} = \bigsqcup\{\bigsqcup\{\langle f\rangle^*\{a\}\mid a\in A\}\mid A\in S\} = \bigsqcup\{\langle f\rangle^*A\mid A\in S\} = \bigsqcup\langle\langle f\rangle^*\rangle S$.

(2)⇒(4). $\uparrow^{\operatorname{Dst} f}J\not\asymp\langle f\rangle\bigsqcup S\Leftrightarrow\bigsqcup S\,[f]\uparrow^{\operatorname{Dst} f}J\Leftrightarrow\exists\mathcal{I}\in S:\mathcal{I}\,[f]\uparrow^{\operatorname{Dst} f}J\Leftrightarrow\exists\mathcal{I}\in S:\uparrow^{\operatorname{Dst} f}J\not\asymp\langle f\rangle\mathcal{I}\Leftrightarrow\uparrow^{\operatorname{Dst} f}J\not\asymp\bigsqcup\langle\langle f\rangle\rangle S$ (used theorem 4.215).

(2)⇒(3), (4)⇒(5), (5)⇒(3), (5)⇒(6). Obvious. □

The following proposition shows that complete funcoids are a direct generalization of pretopological spaces.

Proposition 6.104. To specify a complete funcoid f it is enough to specify $\langle f\rangle^*$ on one-element sets, values of $\langle f\rangle^*$ on one element sets can be specified arbitrarily.

Proof. From the above theorem is clear that knowing $\langle f\rangle^*$ on one-element sets $\langle f\rangle^*$ can be found on every set and then the value of $\langle f\rangle$ can be inferred for every filter.

Choosing arbitrarily the values of $\langle f\rangle^*$ on one-element sets we can define a complete funcoid the following way: $\langle f\rangle^*X=\bigsqcup\{\langle f\rangle^*\{\alpha\}\mid\alpha\in X\}$ for every $X\in\mathscr{P}(\operatorname{Src} f)$. Obviously it is really a complete funcoid. □

Theorem 6.105. A funcoid is principal iff it is both complete and co-complete.

Proof.

⇒. Obvious.

⇐. Let f be both a complete and co-complete funcoid. Consider the relation g defined by that $\uparrow^{\operatorname{Dst} f}\langle g\rangle\{\alpha\}=\langle f\rangle^*\{\alpha\}$ (g is correctly defined because f corresponds to a generalized closure). Because f is a complete funcoid f is the funcoid corresponding to g. □

Theorem 6.106. If $R\in\mathscr{P}\mathsf{FCD}(A;B)$ is a set of (co-)complete funcoids then $\bigsqcup R$ is a (co-)complete funcoid (for every sets A and B).

Proof. It is enough to prove for co-complete funcoids. Let $R\in\mathscr{P}\mathsf{FCD}(A;B)$ be a set of co-complete funcoids. Then for every $X\in\mathscr{P}(\operatorname{Src} f)$

$$\left\langle\bigsqcup R\right\rangle^*X=\bigsqcup\{\langle f\rangle^*X\mid f\in R\}$$

is a principal filter (used theorem 6.37). □

Corollary 6.107. If R is a set of binary relations between sets A and B then $\bigsqcup\left\langle\uparrow^{\mathsf{FCD}(A;B)}\right\rangle R=\uparrow^{\mathsf{FCD}(A;B)}\bigcup R$.

Proof. From two last theorems. □

Theorem 6.108. Filtrators of funcoids are filtered.

Proof. It's enough to prove that every funcoid is representable as an (infinite) meet (on the lattice $\mathsf{FCD}(A;B)$) of some set of principal funcoids.

Let $f\in\mathsf{FCD}(A;B)$, $X\in\mathscr{P}A$, $Y\in\langle f\rangle X$, $g(X;Y)\overset{\text{def}}{=}\uparrow^A X\times^{\mathsf{FCD}}\uparrow^B Y\sqcup\uparrow^A\overline{X}\times^{\mathsf{FCD}}1^{\mathfrak{F}(B)}$. For every $K\in\mathscr{P}A$

$$\langle g(X;Y)\rangle^*K=\langle\uparrow^A X\times^{\mathsf{FCD}}\uparrow^B Y\rangle^*K\sqcup\left\langle\uparrow^A\overline{X}\times^{\mathsf{FCD}}1^{\mathfrak{F}(B)}\right\rangle^*K=\left(\begin{cases}0^{\mathfrak{F}(B)} & \text{if } K=\emptyset\\ \uparrow^B Y & \text{if } \emptyset\neq K\subseteq X\\ 1^{\mathfrak{F}(B)} & \text{if } K\nsubseteq X\end{cases}\right)\sqsupseteq\langle f\rangle^*K;$$

so $g(X;Y)\sqsupseteq f$. For every $X\in\mathscr{P}A$

$$\bigsqcap\{\langle g(X;Y)\rangle^*X\mid Y\in\langle f\rangle^*X\}=\bigsqcap\{\uparrow^B Y\mid Y\in\langle f\rangle^*X\}=\langle f\rangle^*X;$$