Theorem 6.95. Let f be a funcoid.

1. $\mathcal{X}\,[f]\,\mathcal{Y} \Leftrightarrow \exists F \in \operatorname{atoms} f: \mathcal{X}\,[F]\,\mathcal{Y}$ for every $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{Y} \in \mathfrak{F}(\operatorname{Dst} f)$;
2. $\langle f \rangle \mathcal{X} = \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$ for every $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$.

Proof. 1. $\exists F \in \operatorname{atoms} f: \mathcal{X}\,[F]\,\mathcal{Y} \Leftrightarrow \exists a \in \operatorname{atoms}^{\mathfrak{F}(\operatorname{Src} f)}, b \in \operatorname{atoms}^{\mathfrak{F}(\operatorname{Dst} f)}: (a \times^{\mathsf{FCD}} b \not\asymp f \wedge \mathcal{X}\,[a \times^{\mathsf{FCD}} b]\,\mathcal{Y}) \Leftrightarrow \exists a \in \operatorname{atoms}^{\mathfrak{F}(\operatorname{Src} f)}, b \in \operatorname{atoms}^{\mathfrak{F}(\operatorname{Dst} f)}: (a \times^{\mathsf{FCD}} b \not\asymp f \wedge a \times^{\mathsf{FCD}} b \not\asymp \mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y}) \Leftrightarrow \exists F \in \operatorname{atoms} f: (F \not\asymp f \wedge F \not\asymp \mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y}) \Leftrightarrow f \not\asymp \mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y} \Leftrightarrow \mathcal{X}\,[f]\,\mathcal{Y}$.

2. Let $\mathcal{Y} \in \mathfrak{F}(\operatorname{Dst} f)$. Suppose $\mathcal{Y} \not\asymp \langle f \rangle \mathcal{X}$. Then $\mathcal{X}\,[f]\,\mathcal{Y}$; $\exists F \in \operatorname{atoms} f: \mathcal{X}\,[F]\,\mathcal{Y}$; $\exists F \in \operatorname{atoms} f: \mathcal{Y} \not\asymp \langle F \rangle \mathcal{X}$; $\mathcal{Y} \not\asymp \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$. So $\langle f \rangle \mathcal{X} \sqsubseteq \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$. The contrary $\langle f \rangle \mathcal{X} \sqsupseteq \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$ is obvious. □

Problem 6.96. Let A and B be infinite sets. Characterize the set of all coatoms of the lattice $\mathsf{FCD}(A; B)$ of funcoids from A to B. Particularly, is this set empty? Is $\mathsf{FCD}(A; B)$ a coatomic lattice? coatomistic lattice?

6.11 Complete funcoids

Definition 6.97. I will call *co-complete* such a funcoid f that $\langle f \rangle^* X$ is a principal filter for every $X \in \mathscr{P}(\operatorname{Src} f)$.

Obvious 6.98. Funcoid f is co-complete iff $\langle f \rangle \mathcal{X} \in \mathfrak{P}$ for every $\mathcal{X} \in \mathfrak{P}$.

Definition 6.99. I will call *generalized closure* such a function $\alpha \in (\mathscr{P}B)^{\mathscr{P}A}$ (for some sets A, B) that

1. $\alpha \emptyset = \emptyset$;
2. $\forall I, J \in \mathscr{P}A: \alpha(I \cup J) = \alpha I \cup \alpha J$.

Obvious 6.100. A funcoid f is co-complete iff $\langle f \rangle^* = \uparrow^{\operatorname{Dst} f} \circ \alpha$ for a generalized closure α.

Remark 6.101. Thus funcoids can be considered as a generalization of generalized closures. A topological space in Kuratowski sense is the same as reflexive and transitive generalized closure. So topological spaces can be considered as a special case of funcoids.

Definition 6.102. I will call a *complete funcoid* a funcoid whose reverse is co-complete.

Theorem 6.103. The following conditions are equivalent for every funcoid f:

1. funcoid f is complete;
2. $\forall S \in \mathscr{P}\mathfrak{F}(\operatorname{Src} f), J \in \mathscr{P}(\operatorname{Dst} f): (\bigsqcup S\,[f]\,\uparrow^{\operatorname{Dst} f} J \Leftrightarrow \exists \mathcal{I} \in S: \mathcal{I}\,[f]\,\uparrow^{\operatorname{Dst} f} J)$;
3. $\forall S \in \mathscr{P}\mathscr{P}(\operatorname{Src} f), J \in \mathscr{P}(\operatorname{Dst} f): (\bigcup S\,[f]^*\,J \Leftrightarrow \exists I \in S: I\,[f]^*\,J)$;
4. $\forall S \in \mathscr{P}\mathfrak{F}(\operatorname{Src} f): \langle f \rangle \bigsqcup S = \bigsqcup \langle \langle f \rangle \rangle S$;
5. $\forall S \in \mathscr{P}\mathscr{P}(\operatorname{Src} f): \langle f \rangle^* \bigcup S = \bigsqcup \langle \langle f \rangle^* \rangle S$;
6. $\forall A \in \mathscr{P}(\operatorname{Src} f): \langle f \rangle^* A = \bigsqcup \{ \langle f \rangle^* \{a\} \mid a \in A \}$.

Proof.

(3)⇒(1). For every $S \in \mathscr{P}\mathscr{P}(\operatorname{Src} f)$, $J \in \mathscr{P}(\operatorname{Dst} f)$

$$\uparrow^{\operatorname{Src} f} \bigcup S \sqcap \langle f^{-1} \rangle^* J \neq 0^{\mathfrak{F}(\operatorname{Src} f)} \Leftrightarrow \exists I \in S: \uparrow^{\operatorname{Src} f} I \sqcap \langle f^{-1} \rangle^* J \neq 0^{\mathfrak{F}(\operatorname{Src} f)},$$

consequently by proposition 4.215 we have that $\langle f^{-1} \rangle^* J$ is a principal filter.

(1)⇒(2). For every $S \in \mathscr{P}\mathfrak{F}(\operatorname{Src} f)$, $J \in \mathscr{P}(\operatorname{Dst} f)$ we have $\langle f^{-1} \rangle^* J$ is a principal filter, consequently

$$\bigsqcup S \sqcap \langle f^{-1} \rangle^* J \neq 0^{\mathfrak{F}(\operatorname{Src} f)} \Leftrightarrow \exists \mathcal{I} \in S: \mathcal{I} \sqcap \langle f^{-1} \rangle^* J \neq 0^{\mathfrak{F}(\operatorname{Src} f)}.$$