Proof.

$$
\begin{aligned}
g\circ f\not\asymp h & \Leftrightarrow \\
\exists a\in\operatorname{atoms}^{\mathfrak{F}(A)}, c\in\operatorname{atoms}^{\mathfrak{F}(C)}: a\,[(g\circ f)\sqcap h]\,c & \Leftrightarrow \\
\exists a\in\operatorname{atoms}^{\mathfrak{F}(A)}, c\in\operatorname{atoms}^{\mathfrak{F}(C)}: (a\,[g\circ f]\,c\wedge a\,[h]\,c) & \Leftrightarrow \\
\exists a\in\operatorname{atoms}^{\mathfrak{F}(A)}, b\in\operatorname{atoms}^{\mathfrak{F}(B)}, c\in\operatorname{atoms}^{\mathfrak{F}(C)}: (a\,[f]\,b\wedge b\,[g]\,c\wedge a\,[h]\,c) & \Leftrightarrow \\
\exists b\in\operatorname{atoms}^{\mathfrak{F}(B)}, c\in\operatorname{atoms}^{\mathfrak{F}(C)}: (b\,[g]\,c\wedge b\,[h\circ f^{-1}]\,c) & \Leftrightarrow \\
\exists b\in\operatorname{atoms}^{\mathfrak{F}(B)}, c\in\operatorname{atoms}^{\mathfrak{F}(C)}: b\,[g\sqcap (h\circ f^{-1})]\,c & \Leftrightarrow \\
g\not\asymp h\circ f^{-1}. &
\end{aligned}
$$

□

6.9 Direct product of filters

A generalization of Cartesian product of two sets is funcoidal product of two filters:

Definition 6.71. *Funcoidal product* of filters $\mathcal{A}$ and $\mathcal{B}$ is such a funcoid $\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\in\mathsf{FCD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))$ that for every $\mathcal{X}\in\mathfrak{F}(\operatorname{Base}(\mathcal{A}))$, $\mathcal{Y}\in\mathfrak{F}(\operatorname{Base}(\mathcal{B}))$

$$\mathcal{X}\,[\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}]\,\mathcal{Y}\Leftrightarrow\mathcal{X}\not\asymp\mathcal{A}\wedge\mathcal{Y}\not\asymp\mathcal{B}.$$

Proposition 6.72. $\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$ is really a funcoid and

$$\langle\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\rangle\mathcal{X}=\begin{cases}\mathcal{B} & \text{if } \mathcal{X}\not\asymp\mathcal{A}\\ 0^{\mathfrak{F}(\operatorname{Base}(\mathcal{B}))} & \text{if } \mathcal{X}\asymp\mathcal{A}.\end{cases}$$

Proof. Obvious. □

Obvious 6.73. $\uparrow^{\mathsf{FCD}(U;V)}(A\times B)=\uparrow^U A\times^{\mathsf{FCD}}\uparrow^V B$ for sets $A\subseteq U$ and $B\subseteq V$.

Proposition 6.74. $f\sqsubseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\Leftrightarrow\operatorname{dom} f\sqsubseteq\mathcal{A}\wedge\operatorname{im} f\sqsubseteq\mathcal{B}$ for every $f\in\mathsf{FCD}(A;B)$ and $\mathcal{A}\in\mathfrak{F}(A)$, $\mathcal{B}\in\mathfrak{F}(B)$.

Proof. If $f\sqsubseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$ then $\operatorname{dom} f\sqsubseteq\operatorname{dom}(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})\sqsubseteq\mathcal{A}$, $\operatorname{im} f\sqsubseteq\operatorname{im}(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})\sqsubseteq\mathcal{B}$. If $\operatorname{dom} f\sqsubseteq\mathcal{A}\wedge\operatorname{im} f\sqsubseteq\mathcal{B}$ then

$$\forall\mathcal{X}\in\mathfrak{F}(A),\mathcal{Y}\in\mathfrak{F}(B):\left(\mathcal{X}\,[f]\,\mathcal{Y}\Rightarrow\mathcal{X}\sqcap\mathcal{A}\neq 0^{\mathfrak{F}(A)}\wedge\mathcal{Y}\sqcap\mathcal{B}\neq 0^{\mathfrak{F}(B)}\right);$$

consequently $f\sqsubseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$. □

The following theorem gives a formula for calculating an important particular case of a meet on the lattice of funcoids:

Theorem 6.75. $f\sqcap(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})=\operatorname{id}^{\mathsf{FCD}}_{\mathcal{B}}\circ f\circ\operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}}$ for every funcoid f and $\mathcal{A}\in\mathfrak{F}(\operatorname{Src} f)$, $\mathcal{B}\in\mathfrak{F}(\operatorname{Dst} f)$.

Proof. $h\overset{\text{def}}{=}\operatorname{id}^{\mathsf{FCD}}_{\mathcal{B}}\circ f\circ\operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}}$. For every $\mathcal{X}\in\mathfrak{F}(\operatorname{Src} f)$

$$\langle h\rangle\mathcal{X}=\langle\operatorname{id}^{\mathsf{FCD}}_{\mathcal{B}}\rangle\langle f\rangle\langle\operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}}\rangle\mathcal{X}=\mathcal{B}\sqcap\langle f\rangle(\mathcal{A}\sqcap\mathcal{X}).$$

From this, as easy to show, $h\sqsubseteq f$ and $h\sqsubseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$. If $g\sqsubseteq f\wedge g\sqsubseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$ for a $g\in\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)$ then $\operatorname{dom} g\sqsubseteq\mathcal{A}$, $\operatorname{im} g\sqsubseteq\mathcal{B}$,

$$\langle g\rangle\mathcal{X}=\mathcal{B}\sqcap\langle g\rangle(\mathcal{A}\sqcap\mathcal{X})\sqsubseteq\mathcal{B}\sqcap\langle f\rangle(\mathcal{A}\sqcap\mathcal{X})=\langle\operatorname{id}^{\mathsf{FCD}}_{\mathcal{B}}\rangle\langle f\rangle\langle\operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}}\rangle\mathcal{X}=\langle h\rangle\mathcal{X},$$

$g\sqsubseteq h$. So $h=f\sqcap(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})$. □

Corollary 6.76. $f|_{\mathcal{A}}=f\sqcap\left(\mathcal{A}\times^{\mathsf{FCD}}1^{\mathfrak{F}(\operatorname{Dst} f)}\right)$ for every funcoid f and $\mathcal{A}\in\mathfrak{F}(\operatorname{Src} f)$.