Proof.

1. $\operatorname{im}(g \circ f) = \langle g \circ f \rangle 1^{\mathfrak{F}(\operatorname{Src} f)} = \langle g \rangle \langle f \rangle 1^{\mathfrak{F}(\operatorname{Src} f)} = \langle g \rangle \operatorname{im} f = \langle g \rangle (\operatorname{im} f \sqcap \operatorname{dom} g) = \langle g \rangle \operatorname{dom} g = \langle g \rangle 1^{\mathfrak{F}(\operatorname{Src} g)} = \operatorname{im} g$.
2. $\operatorname{dom}(g \circ f) = \operatorname{im}(f^{-1} \circ g^{-1})$ what by proved above is equal to $\operatorname{im} f^{-1}$ that is $\operatorname{dom} f$. □

Lemma 6.63. $\lambda \mathcal{B} \in \mathfrak{F}(B): 1^{\mathfrak{F}} \times^{\mathsf{FCD}} \mathcal{B}$ is an upper adjoint of $\lambda f \in \mathsf{FCD}(A; B): \operatorname{im} f$ (for every sets A, B).

Proof. We need to prove $\operatorname{im} f \sqsubseteq \mathcal{B} \Leftrightarrow f \sqsubseteq 1^{\mathfrak{F}} \times^{\mathsf{FCD}} \mathcal{B}$ what is obvious. □

Corollary 6.64. Image and domain of funcoids preserve joins.

Proof. By properties of Galois connections and duality. □

6.7 Categories of funcoids

I will define two categories, the *category of funcoids* and the *category of funcoid triples*.

The *category of funcoids* is defined as follows:

- Objects are small sets.
- The set of morphisms from a set A to a set B is $\mathsf{FCD}(A; B)$.
- The composition is the composition of funcoids.
- Identity morphism for a set is the identity funcoid for that set.

To show it is really a category is trivial.

The *category of funcoid triples* is defined as follows:

- Objects are filters on small sets.
- The morphisms from a filter $\mathcal{A}$ to a filter $\mathcal{B}$ are triples $(\mathcal{A}; \mathcal{B}; f)$ where $f \in \mathsf{FCD}(\operatorname{Base}(\mathcal{A}); \operatorname{Base}(\mathcal{B}))$ and $\operatorname{dom} f \sqsubseteq \mathcal{A} \wedge \operatorname{im} f \sqsubseteq \mathcal{B}$.
- The composition is defined by the formula $(\mathcal{B}; \mathcal{C}; g) \circ (\mathcal{A}; \mathcal{B}; f) = (\mathcal{A}; \mathcal{C}; g \circ f)$.
- Identity morphism for a filter $\mathcal{A}$ is $\operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}}$.

To prove that it is really a category is trivial.

6.8 Specifying funcoids by functions or relations on atomic filters

Theorem 6.65. For every funcoid f and $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{Y} \in \mathfrak{F}(\operatorname{Dst} f)$

1. $\langle f \rangle \mathcal{X} = \bigsqcup \langle \langle f \rangle \rangle \operatorname{atoms} \mathcal{X}$;
2. $\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \exists x \in \operatorname{atoms} \mathcal{X}, y \in \operatorname{atoms} \mathcal{Y}: x [f] y$.

Proof. 1.

$$\begin{aligned} \mathcal{Y} \sqcap \langle f \rangle \mathcal{X} \neq 0^{\mathfrak{F}(\operatorname{Dst} f)} &\Leftrightarrow \mathcal{X} \sqcap \langle f^{-1} \rangle \mathcal{Y} \neq 0^{\mathfrak{F}(\operatorname{Src} f)} \\ &\Leftrightarrow \exists x \in \operatorname{atoms} \mathcal{X}: x \sqcap \langle f^{-1} \rangle \mathcal{Y} \neq 0^{\mathfrak{F}(\operatorname{Src} f)} \\ &\Leftrightarrow \exists x \in \operatorname{atoms} \mathcal{X}: \mathcal{Y} \sqcap \langle f \rangle x \neq 0^{\mathfrak{F}(\operatorname{Dst} f)}. \end{aligned}$$

$\partial \langle f \rangle \mathcal{X} = \bigsqcup \langle \partial \rangle \langle \langle f \rangle \rangle \operatorname{atoms} \mathcal{X} = \partial \bigsqcup \langle \langle f \rangle \rangle \operatorname{atoms} \mathcal{X}$. So $\langle f \rangle \mathcal{X} = \bigsqcup \langle \langle f \rangle \rangle \operatorname{atoms} \mathcal{X}$ by proposition 4.202.

2. If $\mathcal{X} [f] \mathcal{Y}$, then $\mathcal{Y} \sqcap \langle f \rangle \mathcal{X} \neq 0^{\mathfrak{F}(\operatorname{Dst} f)}$, consequently there exists $y \in \operatorname{atoms} \mathcal{Y}$ such that $y \sqcap \langle f \rangle \mathcal{X} \neq 0^{\mathfrak{F}(\operatorname{Dst} f)}$, $\mathcal{X} [f] y$. Repeating this second time we get that there exists $x \in \operatorname{atoms} \mathcal{X}$ such that $x [f] y$. From this it follows

$$\exists x \in \operatorname{atoms} \mathcal{X}, y \in \operatorname{atoms} \mathcal{Y}: x [f] y.$$