Proof. $\mathcal{X}\,[g\circ f]\,\mathcal{Y}\Leftrightarrow\mathcal{Y}\sqcap\langle g\circ f\rangle\mathcal{X}\neq 0^{\mathfrak{F}(\mathrm{Dst}\,g)}\Leftrightarrow\mathcal{Y}\sqcap\langle g\rangle\langle f\rangle\mathcal{X}\neq 0^{\mathfrak{F}(\mathrm{Dst}\,g)}\Leftrightarrow\langle f\rangle\mathcal{X}\,[g]\,\mathcal{Y}\Leftrightarrow\mathcal{X}([g]\circ\langle f\rangle)\mathcal{Y}$ for every $\mathcal{X}\in\mathfrak{F}(\mathrm{Src}\,f)$, $\mathcal{Y}\in\mathfrak{F}(\mathrm{Dst}\,g)$. $[g\circ f]=[(f^{-1}\circ g^{-1})^{-1}]\Leftrightarrow[f^{-1}\circ g^{-1}]^{-1}=([f^{-1}]\circ\langle g^{-1}\rangle)^{-1}=\langle g^{-1}\rangle^{-1}\circ[f]$. □

Corollary 6.43. $[f]=[\mathrm{id}_{\mathrm{Dst}\,f}]\circ\langle f\rangle$ for every funcoid f.

The following theorem is a variant for funcoids of the statement (which defines compositions of relations) that $x\,(g\circ f)\,z\Leftrightarrow\exists y\colon(x\,f\,y\wedge y\,g\,z)$ for every x and z and every binary relations f and g.

Theorem 6.44. For every sets A, B, C and $f\in\mathsf{FCD}(A;B)$, $g\in\mathsf{FCD}(B;C)$ and $\mathcal{X}\in\mathfrak{F}(A)$, $\mathcal{Z}\in\mathfrak{F}(C)$

$$\mathcal{X}\,[g\circ f]\,\mathcal{Z}\Leftrightarrow\exists y\in\mathrm{atoms}^{\mathfrak{F}(B)}\colon(\mathcal{X}\,[f]\,y\wedge y\,[g]\,\mathcal{Z}).$$

Proof.

$$\begin{aligned}\exists y\in\mathrm{atoms}^{\mathfrak{F}(B)}\colon(\mathcal{X}\,[f]\,y\wedge y\,[g]\,\mathcal{Z}) &\Leftrightarrow\\ \exists y\in\mathrm{atoms}^{\mathfrak{F}(B)}\colon\left(\mathcal{Z}\sqcap\langle g\rangle y\neq 0^{\mathfrak{F}(C)}\wedge y\sqcap\langle f\rangle\mathcal{X}\neq 0^{\mathfrak{F}(B)}\right) &\Leftrightarrow\\ \exists y\in\mathrm{atoms}^{\mathfrak{F}(B)}\colon\left(\mathcal{Z}\sqcap\langle g\rangle y\neq 0^{\mathfrak{F}(C)}\wedge y\sqsubseteq\langle f\rangle\mathcal{X}\right) &\Rightarrow\\ \mathcal{Z}\sqcap\langle g\rangle\langle f\rangle\mathcal{X}\neq 0^{\mathfrak{F}(C)} &\Leftrightarrow\\ \mathcal{X}\,[g\circ f]\,\mathcal{Z}.&\end{aligned}$$

Reversely, if $\mathcal{X}\,[g\circ f]\,\mathcal{Z}$ then $\langle f\rangle\mathcal{X}\,[g]\,\mathcal{Z}$, consequently there exists $y\in\mathrm{atoms}\,\langle f\rangle\mathcal{X}$ such that $y\,[g]\,\mathcal{Z}$; we have $\mathcal{X}\,[f]\,y$. □

Theorem 6.45. For every sets A, B, C

1. $f\circ(g\sqcup h)=f\circ g\sqcup f\circ h$ for $g,h\in\mathsf{FCD}(A;B)$ and $f\in\mathsf{FCD}(B;C)$;
2. $(g\sqcup h)\circ f=g\circ f\sqcup h\circ f$ for $g,h\in\mathsf{FCD}(B;C)$ and $f\in\mathsf{FCD}(A;B)$.

Proof. I will prove only the first equality because the other is analogous.

For every $\mathcal{X}\in\mathfrak{F}(A)$, $\mathcal{Z}\in\mathfrak{F}(C)$

$$\begin{aligned}\mathcal{X}\,[f\circ(g\sqcup h)]\,\mathcal{Z} &\Leftrightarrow \exists y\in\mathrm{atoms}^{\mathfrak{F}(B)}\colon(\mathcal{X}\,[g\sqcup h]\,y\wedge y\,[f]\,\mathcal{Z})\\ &\Leftrightarrow \exists y\in\mathrm{atoms}^{\mathfrak{F}(B)}\colon((\mathcal{X}\,[g]\,y\vee\mathcal{X}\,[h]\,y)\wedge y\,[f]\,\mathcal{Z})\\ &\Leftrightarrow \exists y\in\mathrm{atoms}^{\mathfrak{F}(B)}\colon((\mathcal{X}\,[g]\,y\wedge y\,[f]\,\mathcal{Z})\vee(\mathcal{X}\,[h]\,y\wedge y\,[f]\,\mathcal{Z}))\\ &\Leftrightarrow \exists y\in\mathrm{atoms}^{\mathfrak{F}(B)}\colon(\mathcal{X}\,[g]\,y\wedge y\,[f]\,\mathcal{Z})\vee\exists y\in\mathrm{atoms}^{\mathfrak{F}(B)}\colon(\mathcal{X}\,[h]\,y\wedge y\,[f]\,\mathcal{Z})\\ &\Leftrightarrow \mathcal{X}\,[f\circ g]\,\mathcal{Z}\vee\mathcal{X}\,[f\circ h]\,\mathcal{Z}\\ &\Leftrightarrow \mathcal{X}\,[f\circ g\sqcup f\circ h]\,\mathcal{Z}.\end{aligned}$$

□

Remark 6.46. The above theorem can be proved without atomic filters by the formula $\langle f\circ(g\sqcup h)\rangle\mathcal{X}=\langle f\rangle\langle g\sqcup h\rangle\mathcal{X}=\langle f\rangle(\langle g\rangle\mathcal{X}\sqcup\langle h\rangle\mathcal{X})=\langle f\rangle\langle g\rangle\mathcal{X}\sqcup\langle f\rangle\langle h\rangle\mathcal{X}=\langle f\circ g\rangle\mathcal{X}\sqcup\langle f\circ h\rangle\mathcal{X}=\langle f\circ g\sqcup f\circ h\rangle\mathcal{X}$. [TODO: This may be useful for a pointfree generalization. Preserve old proof for history.]

6.6 Domain and range of a funcoid

Definition 6.47. Let A be a set. The *identity funcoid* $\mathrm{id}^{\mathsf{FCD}(A)}=(A;A;\mathrm{id}_{\mathfrak{F}(A)};\mathrm{id}_{\mathfrak{F}(A)})$.

Obvious 6.48. The identity funcoid is a funcoid.

Definition 6.49. Let A be a set, $\mathcal{A}\in\mathfrak{F}(A)$. The *restricted identity funcoid*

$$\mathrm{id}^{\mathsf{FCD}}_{\mathcal{A}}=(A;A;\mathcal{A}\sqcap;\mathcal{A}\sqcap).$$