2. with co-separable core.

Below it is shown that $\mathsf{FCD}(A;B)$ are complete lattices for every sets A and B. We will apply lattice operations to subsets of such sets without explicitly mentioning $\mathsf{FCD}(A;B)$.

Theorem 6.37. $\mathsf{FCD}(A;B)$ is a complete lattice (for every sets A and B). For every $R\in\mathscr{P}\mathsf{FCD}(A;B)$ and $X\in\mathscr{P}A$, $Y\in\mathscr{P}B$

1. $X\,[\bigsqcup R]^*\,Y\Leftrightarrow\exists f\in R: X\,[f]^*\,Y$;
2. $\langle\bigsqcup R\rangle^*X=\bigsqcup\{\langle f\rangle^*X\mid f\in R\}$.

Proof. Accordingly [26] to prove that it is a complete lattice it's enough to prove existence of all joins.

2. $\alpha X\stackrel{\text{def}}{=}\bigsqcup\{\langle f\rangle^*X\mid f\in R\}$. We have $\alpha\emptyset=0^{\mathfrak{F}(\operatorname{Dst}f)}$;

$$\begin{aligned}\alpha(I\cup J)&=\bigsqcup\{\langle f\rangle^*(I\cup J)\mid f\in R\}\\&=\bigsqcup\{\langle f\rangle^*I\sqcup\langle f\rangle^*J\mid f\in R\}\\&=\bigsqcup\{\langle f\rangle^*I\mid f\in R\}\sqcup\bigsqcup\{\langle f\rangle^*J\mid f\in R\}\\&=\alpha I\sqcup\alpha J.\end{aligned}$$

So $\langle h\rangle^*=\alpha$ for some funcoid h. Obviously

$$\forall f\in R: h\sqsupseteq f. \tag{6.5}$$

And h is the least funcoid for which holds the condition (6.5). So $h=\bigsqcup R$.

1. $X\,[\bigsqcup R]^*\,Y\Leftrightarrow\uparrow^{\operatorname{Dst}f}Y\sqcap\langle\bigsqcup R\rangle^*X\neq 0^{\mathfrak{F}(\operatorname{Dst}f)}\Leftrightarrow\uparrow^{\operatorname{Dst}f}Y\sqcap\bigsqcup\{\langle f\rangle^*X\mid f\in R\}\neq 0^{\mathfrak{F}(\operatorname{Dst}f)}\Leftrightarrow\exists f\in R:\uparrow^{\operatorname{Dst}f}Y\sqcap\langle f\rangle^*X\neq 0^{\mathfrak{F}(\operatorname{Dst}f)}\Leftrightarrow\exists f\in R: X\,[f]^*\,Y$ (used proposition 4.194). □

In the next theorem, compared to the previous one, the class of infinite joins is replaced with lesser class of finite joins and simultaneously class of sets is changed to more wide class of filters.

Theorem 6.38. For every $f,g\in\mathsf{FCD}(A;B)$ and $\mathcal{X}\in\mathfrak{F}(A)$ (for every sets A, B)

1. $\langle f\sqcup g\rangle\mathcal{X}=\langle f\rangle\mathcal{X}\sqcup\langle g\rangle\mathcal{X}$;
2. $[f\sqcup g]=[f]\cup[g]$.

Proof. 1. Let $\alpha\mathcal{X}\stackrel{\text{def}}{=}\langle f\rangle\mathcal{X}\sqcup\langle g\rangle\mathcal{X}$; $\beta\mathcal{Y}\stackrel{\text{def}}{=}\langle f^{-1}\rangle\mathcal{Y}\sqcup\langle g^{-1}\rangle\mathcal{Y}$ for every $\mathcal{X}\in\mathfrak{F}(A)$, $\mathcal{Y}\in\mathfrak{F}(B)$. Then

$$\begin{aligned}\mathcal{Y}\sqcap\alpha\mathcal{X}\neq 0^{\mathfrak{F}(B)}&\Leftrightarrow\mathcal{Y}\sqcap\langle f\rangle\mathcal{X}\neq 0^{\mathfrak{F}(B)}\vee\mathcal{Y}\sqcap\langle g\rangle\mathcal{X}\neq 0^{\mathfrak{F}(B)}\\&\Leftrightarrow\mathcal{X}\sqcap\langle f^{-1}\rangle\mathcal{Y}\neq 0^{\mathfrak{F}(A)}\vee\mathcal{X}\sqcap\langle g^{-1}\rangle\mathcal{Y}\neq 0^{\mathfrak{F}(A)}\\&\Leftrightarrow\mathcal{X}\sqcap\beta\mathcal{Y}\neq 0^{\mathfrak{F}(A)}.\end{aligned}$$

So $h=(A;B;\alpha;\beta)$ is a funcoid. Obviously $h\sqsupseteq f$ and $h\sqsupseteq g$. If $p\sqsupseteq f$ and $p\sqsupseteq g$ for some funcoid p then $\langle p\rangle\mathcal{X}\sqsupseteq\langle f\rangle\mathcal{X}\sqcup\langle g\rangle\mathcal{X}=\langle h\rangle\mathcal{X}$ that is $p\sqsupseteq h$. So $f\sqcup g=h$.

2. $\mathcal{X}\,[f\sqcup g]\,\mathcal{Y}\Leftrightarrow\mathcal{Y}\sqcap\langle f\sqcup g\rangle\mathcal{X}\neq 0^{\mathfrak{F}(B)}\Leftrightarrow\mathcal{Y}\sqcap(\langle f\rangle\mathcal{X}\sqcup\langle g\rangle\mathcal{X})\neq 0^{\mathfrak{F}(B)}\Leftrightarrow\mathcal{Y}\sqcap\langle f\rangle\mathcal{X}\neq 0^{\mathfrak{F}(B)}\vee\mathcal{Y}\sqcap\langle g\rangle\mathcal{X}\neq 0^{\mathfrak{F}(B)}\Leftrightarrow\mathcal{X}\,[f]\,\mathcal{Y}\vee\mathcal{X}\,[g]\,\mathcal{Y}$ for every $\mathcal{X}\in\mathfrak{F}(A)$, $\mathcal{Y}\in\mathfrak{F}(B)$. □

Definition 6.39. $\operatorname{GR}f\stackrel{\text{def}}{=}\left\{F\in\mathscr{P}(\operatorname{Src}f\times\operatorname{Dst}f)\mid\uparrow^{\mathsf{FCD}(\operatorname{Src}f;\operatorname{Dst}f)}F\sqsupseteq f\right\}$.

Definition 6.40. $\operatorname{xyGR}f\stackrel{\text{def}}{=}\{(\operatorname{Src}f;\operatorname{Dst}f;F)\mid F\in\operatorname{GR}f\}$.

Remark 6.41. $\operatorname{xyGR}f$ is a set of Rel-morphisms.

6.5 More on composition of funcoids

Proposition 6.42. $[g\circ f]=[g]\circ\langle f\rangle=\langle g^{-1}\rangle^{-1}\circ[f]$ for every composable funcoids f and g.