$\forall X \in \mathscr{P}A, Y \in \mathscr{P}B$: $\left(\uparrow^B Y \sqcap \langle f\rangle \uparrow^A X \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \uparrow^A X\ [f]\ \uparrow^B Y \Leftrightarrow X\ \delta\ Y \Leftrightarrow \uparrow^B Y \sqcap \alpha X \neq 0^{\mathfrak{F}(B)}\right)$, consequently $\forall X \in \mathscr{P}A: \alpha X = \langle f\rangle \uparrow^A X = \langle f\rangle^* X$. □

Note that by the last theorem to every proximity δ corresponds a unique funcoid. So funcoids are a generalization of (quasi-)proximity structures. Reverse funcoids can be considered as a generalization of conjugate quasi-proximity.

Definition 6.29. Any (multivalued) function $F: A \to B$ corresponds to a funcoid $\uparrow^{\mathsf{FCD}(A;B)} F \in \mathsf{FCD}(A;B)$, where by definition $\langle \uparrow^{\mathsf{FCD}(A;B)} F\rangle \mathcal{X} = \bigsqcap \langle \uparrow^B\rangle \langle\langle F\rangle\rangle \mathcal{X}$ for every $\mathcal{X} \in \mathfrak{F}(A)$.

Using the last theorem it is easy to show that this definition is monovalued and does not contradict to former stuff. (Take $\alpha = \uparrow^B \circ \langle F\rangle$.)

Definition 6.30. $\uparrow^{\mathsf{FCD}} f \overset{\text{def}}{=} \left(\operatorname{Src} f; \operatorname{Dst} f; \uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} \operatorname{GR} f\right)$ for every Rel-morphism f.

Definition 6.31. Funcoids corresponding to a binary relation (= multivalued function) are called *principal funcoids*.

We may equate principal funcoids with corresponding binary relations by the method of appendix B in [29]. This is useful for describing relationships of funcoids and binary relations, such as for the formulas of continuous functions and continuous funcoids (see below).

Theorem 6.32. If S is a generalized filter base on $\operatorname{Src} f$ then $\langle f\rangle \bigsqcap S = \bigsqcap \langle\langle f\rangle\rangle S$ for every funcoid f.

Proof. $\langle f\rangle \bigsqcap S \sqsubseteq \langle f\rangle X$ for every $X \in S$ and thus $\langle f\rangle \bigsqcap S \sqsubseteq \bigsqcap \langle\langle f\rangle\rangle S$.

By properties of generalized filter bases:

$\langle f\rangle \bigsqcap S = \bigsqcap \langle\langle f\rangle^*\rangle \bigsqcap S =$

$\bigsqcap \langle\langle f\rangle^*\rangle \{X \mid \exists \mathcal{P} \in S: X \in \mathcal{P}\} = \bigsqcap \{\langle f\rangle^* X \mid \exists \mathcal{P} \in S: X \in \mathcal{P}\} \sqsupseteq \bigsqcap \{\langle f\rangle \mathcal{P} \mid \mathcal{P} \in S\} = \bigsqcap \langle\langle f\rangle\rangle S$. □

6.4 Lattices of funcoids

Definition 6.33. $f \sqsubseteq g \overset{\text{def}}{=} [f] \subseteq [g]$ for $f, g \in \mathsf{FCD}$.

Thus every $\mathsf{FCD}(A;B)$ is a poset. (It's taken into account that $[f] \neq [g]$ when $f \neq g$.)

We will consider filtrators (*filtrators of funcoids*) whose base is $\mathsf{FCD}(A;B)$ and whose core are principal funcoids from A to B.

Lemma 6.34. $\langle f\rangle^* X = \bigsqcap \{\langle F\rangle^* X \mid F \in \operatorname{up} f\}$ for every funcoid f and set $X \in \mathscr{P}(\operatorname{Src} f)$.

Proof. Obviously $\langle f\rangle^* X \sqsubseteq \bigsqcap \{\langle F\rangle^* X \mid F \in \operatorname{up} f\}$.

Let $B \in \langle f\rangle^* X$. Let $F_B = X \times B \cup ((\operatorname{Src} f) \setminus X) \times \operatorname{Dst} f$.

$\langle F_B\rangle X = B$.

Let $P \in \mathscr{P}(\operatorname{Src} f)$. We have $\emptyset \neq P \subseteq X \Rightarrow \uparrow^{\operatorname{Dst} f} \langle F_B\rangle P = \uparrow^{\operatorname{Dst} f} B \sqsupseteq \langle f\rangle^* P$ and $\emptyset \neq P \nsubseteq X \Rightarrow \uparrow^{\operatorname{Dst} f} \langle F_B\rangle P = \uparrow^{\operatorname{Dst} f} \operatorname{Dst} f \sqsupseteq \langle f\rangle^* P$. Thus $\uparrow^{\operatorname{Dst} f} \langle F_B\rangle P \sqsupseteq \langle f\rangle^* P$ for every P and so $\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} F_B \sqsupseteq f$ that is $F_B \in \operatorname{up} f$.

Thus $\forall B \in \langle f\rangle^* X: B \in \bigsqcap \{\langle F\rangle^* X \mid F \in \operatorname{up} f\}$ because $B \in \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} F_B\right\rangle^* X$.

So $\bigsqcap \{\langle F\rangle^* X \mid F \in \operatorname{up} f\} \sqsubseteq \langle f\rangle^* X$. □

Theorem 6.35. $\langle f\rangle \mathcal{X} = \bigsqcap \{\langle F\rangle \mathcal{X} \mid F \in \operatorname{up} f\}$ for every funcoid f and $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$.

Proof. $\bigsqcap \{\langle F\rangle \mathcal{X} \mid F \in \operatorname{up} f\} = \bigsqcap \{\bigsqcap \langle\langle F\rangle^*\rangle \mathcal{X} \mid F \in \operatorname{up} f\} = \bigsqcap \{\bigsqcap \{\langle F\rangle^* X \mid X \in \mathcal{X}\} \mid F \in \operatorname{up} f\} = \bigsqcap \{\bigsqcap \{\langle F\rangle^* X \mid F \in \operatorname{up} f\} \mid X \in \mathcal{X}\} = \bigsqcap \{\langle f\rangle^* X \mid X \in \mathcal{X}\} = \langle f\rangle \mathcal{X}$ (the lemma used). □

Conjecture 6.36. Every filtrator of funcoids is: [TODO: Solved. See rewrite-plan.pdf]

1. with separable core;