Theorem 6.28. Fix sets A and B. Let $L_F = \lambda f \in \mathsf{FCD}(A;B): \langle f \rangle^*$ and $L_R = \lambda f \in \mathsf{FCD}(A;B): [f]^*$.

1. L_F is a bijection from the set $\mathsf{FCD}(A;B)$ to the set of functions $\alpha \in \mathfrak{F}(B)^{\mathscr{P}A}$ that obey the conditions (for every $I, J \in \mathscr{P}A$)

$$\alpha\emptyset = 0^{\mathfrak{F}(B)}, \quad \alpha(I \cup J) = \alpha I \sqcup \alpha J. \tag{6.1}$$

For such α it holds (for every $\mathcal{X} \in \mathfrak{F}(A)$)

$$\langle L_F{}^{-1} \alpha \rangle \mathcal{X} = \bigsqcap \langle \alpha \rangle \mathcal{X}. \tag{6.2}$$

2. L_R is a bijection from the set $\mathsf{FCD}(A;B)$ to the set of binary relations $\delta \in \mathscr{P}(\mathscr{P}A \times \mathscr{P}B)$ that obey the conditions

$$\begin{array}{l}\neg(I\,\delta\,\emptyset),\ \ I \cup J\,\delta\,K \Leftrightarrow I\,\delta\,K \vee J\,\delta\,K \ \ \text{(for every } I, J \in \mathscr{P}A,\ K \in \mathscr{P}B),\\ \neg(\emptyset\,\delta\,I),\ \ K\,\delta\,I \cup J \Leftrightarrow K\,\delta\,I \vee K\,\delta\,J \ \ \text{(for every } I, J \in \mathscr{P}B,\ K \in \mathscr{P}A).\end{array} \tag{6.3}$$

For such δ it holds (for every $\mathcal{X} \in \mathfrak{F}(A)$, $\mathcal{Y} \in \mathfrak{F}(B)$)

$$\mathcal{X}\,[L_R{}^{-1}\delta]\,\mathcal{Y} \Leftrightarrow \forall X \in \mathcal{X}, Y \in \mathcal{Y}: X\,\delta\,Y. \tag{6.4}$$

Proof. Injectivity of L_F and L_R, formulas (6.2) (for $\alpha \in \operatorname{im} L_F$) and (6.4) (for $\delta \in \operatorname{im} L_R$), formulas (6.1) and (6.3) follow from two previous theorems. The only thing remained to prove is that for every α and δ that obey the above conditions a corresponding funcoid f exists.

2. Let define $\alpha \in \mathfrak{F}(B)^{\mathscr{P}A}$ by the formula $\partial(\alpha X) = \{Y \in \mathscr{P}B \mid X\,\delta\,Y\}$ for every $X \in \mathscr{P}A$. (It is obvious that $\{Y \in \mathscr{P}B \mid X\,\delta\,Y\}$ is a free star.) Analogously it can be defined $\beta \in \mathfrak{F}(A)^{\mathscr{P}B}$ by the formula $\partial(\beta Y) = \{X \in \mathscr{P}A \mid X\,\delta\,Y\}$. Let's continue α and β to $\alpha' \in \mathfrak{F}(B)^{\mathfrak{F}(A)}$ and $\beta' \in \mathfrak{F}(A)^{\mathfrak{F}(B)}$ by the formulas

$$\alpha'\,\mathcal{X} = \bigsqcap \langle \alpha \rangle \mathcal{X} \quad \text{and} \quad \beta'\,\mathcal{Y} = \bigsqcap \langle \beta \rangle \mathcal{Y}$$

and δ to $\delta' \in \mathscr{P}(\mathfrak{F}(A) \times \mathfrak{F}(B))$ by the formula

$$\mathcal{X}\,\delta'\,\mathcal{Y} \Leftrightarrow \forall X \in \mathcal{X}, Y \in \mathcal{Y}: X\,\delta\,Y.$$

$\mathcal{Y} \sqcap \alpha'\,\mathcal{X} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{Y} \sqcap \bigsqcap \langle \alpha \rangle \mathcal{X} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \bigsqcap \langle \mathcal{Y} \sqcap \rangle \langle \alpha \rangle \mathcal{X} \neq 0^{\mathfrak{F}(B)}$. Let's prove that

$$W = \langle \mathcal{Y} \sqcap \rangle \langle \alpha \rangle \mathcal{X}$$

is a generalized filter base: To prove it is enough to show that $\langle \alpha \rangle \mathcal{X}$ is a generalized filter base. If $\mathcal{A}, \mathcal{B} \in \langle \alpha \rangle \mathcal{X}$ then exist $X_1, X_2 \in \mathcal{X}$ such that $\mathcal{A} = \alpha X_1$, $\mathcal{B} = \alpha X_2$.

Then $\alpha(X_1 \cap X_2) \in \langle \alpha \rangle \mathcal{X}$. So $\langle \alpha \rangle \mathcal{X}$ is a generalized filter base and thus W is a generalized filter base.

By properties of generalized filter bases, $\bigsqcap \langle \mathcal{Y} \sqcap \rangle \langle \alpha \rangle \mathcal{X} \neq 0^{\mathfrak{F}(B)}$ is equivalent to

$$\forall X \in \mathcal{X}: \mathcal{Y} \sqcap \alpha X \neq 0^{\mathfrak{F}(B)},$$

what is equivalent to $\forall X \in \mathcal{X}, Y \in \mathcal{Y}: {\uparrow^B} Y \sqcap \alpha X \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \forall X \in \mathcal{X}, Y \in \mathcal{Y}: Y \in \partial(\alpha X) \Leftrightarrow \forall X \in \mathcal{X}, Y \in \mathcal{Y}: X\,\delta\,Y$. Combining the equivalencies we get $\mathcal{Y} \sqcap \alpha'\,\mathcal{X} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{X}\,\delta'\,\mathcal{Y}$. Analogously $\mathcal{X} \sqcap \beta'\,\mathcal{Y} \neq 0^{\mathfrak{F}(A)} \Leftrightarrow \mathcal{X}\,\delta'\,\mathcal{Y}$. So $\mathcal{Y} \sqcap \alpha'\,\mathcal{X} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{X} \sqcap \beta'\,\mathcal{Y} \neq 0^{\mathfrak{F}(A)}$, that is $(A; B; \alpha'; \beta')$ is a funcoid. From the formula $\mathcal{Y} \sqcap \alpha'\,\mathcal{X} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{X}\,\delta'\,\mathcal{Y}$ it follows that

$$X\,[(A; B; \alpha'; \beta')]^*\,Y \Leftrightarrow {\uparrow^B} Y \sqcap \alpha'\,{\uparrow^A} X \neq 0^{\mathfrak{F}(B)} \Leftrightarrow {\uparrow^A} X\,\delta'\,{\uparrow^B} Y \Leftrightarrow X\,\delta\,Y.$$

1. Let define the relation $\delta \in \mathscr{P}(\mathscr{P}A \times \mathscr{P}B)$ by the formula $X\,\delta\,Y \Leftrightarrow {\uparrow^B} Y \sqcap \alpha X \neq 0^{\mathfrak{F}(B)}$.

That $\neg(I\,\delta\,\emptyset)$ and $\neg(\emptyset\,\delta\,I)$ is obvious. We have $I \cup J\,\delta\,K \Leftrightarrow {\uparrow^B} K \sqcap \alpha(I \cup J) \neq 0^{\mathfrak{F}(B)} \Leftrightarrow {\uparrow^B} K \sqcap (\alpha I \sqcup \alpha J) \neq 0^{\mathfrak{F}(B)} \Leftrightarrow {\uparrow^B} K \sqcap \alpha I \neq 0^{\mathfrak{F}(B)} \vee {\uparrow^B} K \sqcap \alpha J \neq 0^{\mathfrak{F}(B)} \Leftrightarrow I\,\delta\,K \vee J\,\delta\,K$ and

$K\,\delta\,I \cup J \Leftrightarrow {\uparrow^B}(I \cup J) \sqcap \alpha K \neq 0^{\mathfrak{F}(B)} \Leftrightarrow ({\uparrow^B} I \sqcup {\uparrow^B} J) \sqcap \alpha K \neq 0^{\mathfrak{F}(B)} \Leftrightarrow {\uparrow^B} I \sqcap \alpha K \neq 0^{\mathfrak{F}(B)} \vee {\uparrow^B} J \sqcap \alpha K \neq 0^{\mathfrak{F}(B)} \Leftrightarrow K\,\delta\,I \vee K\,\delta\,J$.

That is the formulas (6.3) are true.

Accordingly to the above there exists a funcoid f such that

$$\mathcal{X}\,[f]\,\mathcal{Y} \Leftrightarrow \forall X \in \mathcal{X}, Y \in \mathcal{Y}: X\,\delta\,Y.$$