6.3 Funcoid as continuation

Let f be a funcoid.

Definition 6.21. $\langle f\rangle^{*}$ is the function $\mathscr{P}(\operatorname{Src} f)\rightarrow\mathfrak{F}(\operatorname{Dst} f)$ defined by the formula

$$\langle f\rangle^{*}X=\langle f\rangle\uparrow^{\operatorname{Src} f}X.$$

Definition 6.22. $[f]^{*}$ is the relation between $\mathscr{P}(\operatorname{Src} f)$ and $\mathscr{P}(\operatorname{Dst} f)$ defined by the formula

$$X\,[f]^{*}\,Y\Leftrightarrow\uparrow^{\operatorname{Src} f}X\,[f]\uparrow^{\operatorname{Dst}}Y.$$

Obvious 6.23.

1. $\langle f\rangle^{*}=\langle f\rangle\circ\uparrow^{\operatorname{Src} f}$;
2. $[f]^{*}=(\uparrow^{\operatorname{Dst}})^{-1}\circ[f]\circ\uparrow^{\operatorname{Src} f}$.

Obvious 6.24. $\langle g\rangle\langle f\rangle^{*}X=\langle g\circ f\rangle^{*}X$ for every $X\in\mathscr{P}(\operatorname{Src} f)$.

Theorem 6.25. For every funcoid f and $\mathcal{X}\in\mathfrak{F}(\operatorname{Src} f)$, $\mathcal{Y}\in\mathfrak{F}(\operatorname{Dst} f)$

1. $\langle f\rangle\mathcal{X}=\bigsqcap\langle\langle f\rangle^{*}\rangle\mathcal{X}$;
2. $\mathcal{X}\,[f]\,\mathcal{Y}\Leftrightarrow\forall X\in\mathcal{X},Y\in\mathcal{Y}:X\,[f]^{*}\,Y$.

Proof.

2. $\mathcal{X}\,[f]\,\mathcal{Y}\Leftrightarrow\mathcal{Y}\sqcap\langle f\rangle\mathcal{X}\neq 0^{\mathfrak{F}(\operatorname{Dst} f)}\Leftrightarrow\forall Y\in\mathcal{Y}:\uparrow^{\operatorname{Dst} f}Y\sqcap\langle f\rangle\mathcal{X}\neq 0^{\mathfrak{F}(\operatorname{Dst} f)}\Leftrightarrow\forall Y\in\mathcal{Y}:\mathcal{X}\,[f]\uparrow^{\operatorname{Dst} f}Y$. Analogously $\mathcal{X}\,[f]\,\mathcal{Y}\Leftrightarrow\forall X\in\mathcal{X}:\uparrow^{\operatorname{Src} f}X\,[f]\,\mathcal{Y}$. Combining these two equivalences we get

$$\mathcal{X}\,[f]\,\mathcal{Y}\Leftrightarrow\forall X\in\mathcal{X},Y\in\mathcal{Y}:\uparrow^{\operatorname{Src} f}X\,[f]\uparrow^{\operatorname{Dst} f}Y\Leftrightarrow\forall X\in\mathcal{X},Y\in\mathcal{Y}:X\,[f]^{*}\,Y.$$

1. $\mathcal{Y}\sqcap\langle f\rangle\mathcal{X}\neq 0^{\mathfrak{F}(\operatorname{Dst} f)}\Leftrightarrow\mathcal{X}\,[f]\,\mathcal{Y}\Leftrightarrow\forall X\in\mathcal{X}:\uparrow^{\operatorname{Src} f}X\,[f]\,\mathcal{Y}\Leftrightarrow\forall X\in\mathcal{X}:\mathcal{Y}\sqcap\langle f\rangle^{*}X\neq 0^{\mathfrak{F}(\operatorname{Dst} f)}$.

Let's denote $W=\{\mathcal{Y}\sqcap\langle f\rangle^{*}X\mid X\in\mathcal{X}\}$. We will prove that W is a generalized filter base. To prove this it is enough to show that $V=\{\langle f\rangle^{*}X\mid X\in\mathcal{X}\}$ is a generalized filter base.

Let $\mathcal{P},\mathcal{Q}\in V$. Then $\mathcal{P}=\langle f\rangle^{*}A$, $\mathcal{Q}=\langle f\rangle^{*}B$ where $A,B\in\mathcal{X}$; $A\cap B\in\mathcal{X}$ and $\mathcal{R}\sqsubseteq\mathcal{P}\sqcap\mathcal{Q}$ for $\mathcal{R}=\langle f\rangle^{*}(A\cap B)\in V$. So V is a generalized filter base and thus W is a generalized filter base.

$0^{\mathfrak{F}(\operatorname{Dst} f)}\notin W\Leftrightarrow\bigsqcap W\neq 0^{\mathfrak{F}(\operatorname{Dst} f)}$ by properties of generalized filter bases. That is

$$\forall X\in\mathcal{X}:\mathcal{Y}\sqcap\langle f\rangle^{*}X\neq 0^{\mathfrak{F}(\operatorname{Dst} f)}\Leftrightarrow\mathcal{Y}\sqcap\bigsqcap\langle\langle f\rangle^{*}\rangle\mathcal{X}\neq 0^{\mathfrak{F}(\operatorname{Dst} f)}.$$

Comparing with the above, $\mathcal{Y}\sqcap\langle f\rangle\mathcal{X}\neq 0^{\mathfrak{F}(\operatorname{Dst} f)}\Leftrightarrow\mathcal{Y}\sqcap\bigsqcap\langle\langle f\rangle^{*}\rangle\mathcal{X}\neq 0^{\mathfrak{F}(\operatorname{Dst} f)}$. So $\langle f\rangle\mathcal{X}=\bigsqcap\langle\langle f\rangle^{*}\rangle\mathcal{X}$ because the lattice of filters is separable. □

Corollary 6.26. Let f be a funcoid.

1. The value of f can be restored knowing $\langle f\rangle^{*}$.
2. The value of f can be restored knowing $[f]^{*}$.

Proposition 6.27. For every $f\in\mathsf{FCD}(A;B)$ we have (for every $I,J\in\mathscr{P}A$)

$$\langle f\rangle^{*}\emptyset=0^{\mathfrak{F}(B)},\quad\langle f\rangle^{*}(I\cup J)=\langle f\rangle^{*}I\sqcup\langle f\rangle^{*}J$$

and

$$\neg(I\,[f]^{*}\,\emptyset),\quad I\cup J\,[f]^{*}\,K\Leftrightarrow I\,[f]^{*}\,K\vee J\,[f]^{*}\,K\quad\text{(for every } I,J\in\mathscr{P}A,\ K\in\mathscr{P}B\text{)},$$
$$\neg(\emptyset\,[f]^{*}\,I),\quad K\,[f]^{*}\,I\cup J\Leftrightarrow K\,[f]^{*}\,I\vee K\,[f]^{*}\,J\quad\text{(for every } I,J\in\mathscr{P}B,\ K\in\mathscr{P}A\text{)}.$$

Proof. $\langle f\rangle^{*}\emptyset=\langle f\rangle\uparrow^{A}\emptyset=\langle f\rangle 0^{\mathfrak{F}(A)}=0^{\mathfrak{F}(B)}$; $\langle f\rangle^{*}(I\cup J)=\langle f\rangle\uparrow^{A}(I\cup J)=\langle f\rangle\uparrow^{A}I\sqcup\langle f\rangle\uparrow^{A}J=\langle f\rangle^{*}I\sqcup\langle f\rangle^{*}J$.

$I\,[f]^{*}\,\emptyset\Leftrightarrow 0^{\mathfrak{F}(B)}\not\asymp\langle f\rangle\uparrow^{A}I\Leftrightarrow 0$; $I\cup J\,[f]^{*}\,K\Leftrightarrow\uparrow^{A}(I\cup J)\,[f]\uparrow^{B}K\Leftrightarrow\uparrow^{B}K\not\asymp\langle f\rangle\uparrow^{A}(I\cup J)\Leftrightarrow\uparrow^{B}K\not\asymp\langle f\rangle^{*}(I\cup J)\Leftrightarrow\uparrow^{B}K\not\asymp\langle f\rangle^{*}I\sqcup\langle f\rangle^{*}J\Leftrightarrow\uparrow^{B}K\not\asymp\langle f\rangle^{*}I\vee\uparrow^{B}K\not\asymp\langle f\rangle^{*}J\Leftrightarrow I\,[f]^{*}\,K\vee J\,[f]^{*}\,K$.

The rest follows from symmetry. □