Composition of binary relations (i.e. of principal funcoids) complies with the formulas:

$$\langle g \circ f \rangle = \langle g \rangle \circ \langle f \rangle \quad \text{and} \quad \langle (g \circ f)^{-1} \rangle = \langle f^{-1} \rangle \circ \langle g^{-1} \rangle .$$

By the same formulas we can define composition of every two funcoids. Funcoids with this composition form a category (*the category of funcoids*).

Also funcoids can be reversed (like reversal of X and Y in a binary relation) by the formula $(\alpha; \beta)^{-1} = (\beta; \alpha)$. In the particular case if μ is a proximity we have $\mu^{-1} = \mu$ because proximities are symmetric.

Funcoids behave similarly to (multivalued) functions but acting on filters instead of acting on sets. Below these will be defined domain and image of a funcoid (the domain and the image of a funcoid are filters).

6.2 Basic definitions

Definition 6.1. Let us call a *funcoid* from a set A to a set B a quadruple $(A; B; \alpha; \beta)$ where $\alpha \in \mathfrak{F}(B)^{\mathfrak{F}(A)}$, $\beta \in \mathfrak{F}(A)^{\mathfrak{F}(B)}$ such that

$$\forall \mathcal{X} \in \mathfrak{F}(A), \mathcal{Y} \in \mathfrak{F}(B): (\mathcal{Y} \not\asymp \alpha \mathcal{X} \Leftrightarrow \mathcal{X} \not\asymp \beta \mathcal{Y}).$$

Further we will assume that all funcoids in consideration are small without mentioning it explicitly.

Definition 6.2. *Source* and *destination* of every funcoid $(A; B; \alpha; \beta)$ are defined as:

$$\operatorname{Src}(A; B; \alpha; \beta) = A \quad \text{and} \quad \operatorname{Dst}(A; B; \alpha; \beta) = B.$$

I will denote $\mathsf{FCD}(A; B)$ the set of funcoids from A to B.

I will denote FCD the set of all funcoids (for small sets).

Definition 6.3. $\langle (A; B; \alpha; \beta) \rangle \overset{\text{def}}{=} \alpha$ for a funcoid $(A; B; \alpha; \beta)$.

Definition 6.4. The *reverse* funcoid $(A; B; \alpha; \beta)^{-1} = (B; A; \beta; \alpha)$ for a funcoid $(A; B; \alpha; \beta)$.

Note 6.5. The reverse funcoid is *not* an inverse in the sense of group theory or category theory.

Proposition 6.6. If f is a funcoid then f^{-1} is also a funcoid.

Proof. It follows from symmetry in the definition of funcoid. □

Obvious 6.7. $(f^{-1})^{-1} = f$ for a funcoid f.

Definition 6.8. The relation $[f] \in \mathscr{P}(\mathfrak{F}(\operatorname{Src} f) \times \mathfrak{F}(\operatorname{Dst} f))$ is defined (for every funcoid f and $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{Y} \in \mathfrak{F}(\operatorname{Dst} f)$) by the formula $\mathcal{X} \, [f] \, \mathcal{Y} \Leftrightarrow \mathcal{Y} \not\asymp \langle f \rangle \mathcal{X}$.

Obvious 6.9. $\mathcal{X} \, [f] \, \mathcal{Y} \Leftrightarrow \mathcal{Y} \not\asymp \langle f \rangle \mathcal{X} \Leftrightarrow \mathcal{X} \not\asymp \langle f^{-1} \rangle \mathcal{Y}$ for every funcoid f and $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{Y} \in \mathfrak{F}(\operatorname{Dst} f)$.

Obvious 6.10. $[f^{-1}] = [f]^{-1}$ for a funcoid f.

Theorem 6.11. Let A, B be sets.

1. For given value of $\langle f \rangle \in \mathfrak{F}(B)^{\mathfrak{F}(A)}$ there exists no more than one funcoid $f \in \mathsf{FCD}(A; B)$.
2. For given value of $[f] \in \mathscr{P}(\mathfrak{F}(A) \times \mathfrak{F}(B))$ there exists no more than one funcoid $f \in \mathsf{FCD}(A; B)$.

Proof. Let $f, g \in \mathsf{FCD}(A; B)$.

Obviously, $\langle f \rangle = \langle g \rangle \Rightarrow [f] = [g]$ and $\langle f^{-1} \rangle = \langle g^{-1} \rangle \Rightarrow [f] = [g]$. So it's enough to prove that $[f] = [g] \Rightarrow \langle f \rangle = \langle g \rangle$.