Chapter 6
Funcoids

In this chapter (and several following chapters) the word *filter* will refer to a filter on a set (rather than a filter on an arbitrary poset).

6.1 Informal introduction into funcoids

Funcoids are a generalization of proximity spaces and a generalization of pretopological spaces. Also funcoids are a generalization of binary relations.

That funcoids are a common generalization of "spaces" (proximity spaces, (pre)topological spaces) and binary relations (including monovalued functions) makes them smart for describing properties of functions in regard of spaces. For example the statement "f is a continuous function from a space μ to a space ν" can be described in terms of funcoids as the formula $f\circ\mu\sqsubseteq\nu\circ f$ (see below for details).

Most naturally funcoids appear as a generalization of proximity spaces.

Let δ be a proximity that is certain binary relation so that $A\;\delta\;B$ is defined for every sets A and B. We will extend it from sets to filters by the formula:

$$\mathcal{A}\,\delta'\,\mathcal{B}\Leftrightarrow\forall A\in\mathcal{A},B\in\mathcal{B}\colon A\,\delta\,B.$$

Then (as it will be proved below) there exist two functions $\alpha,\beta\in\mathfrak{F}^{\mathfrak{F}}$ such that

$$\mathcal{A}\,\delta'\,\mathcal{B}\Leftrightarrow\mathcal{B}\sqcap\alpha\mathcal{A}\neq 0^{\mathfrak{F}}\Leftrightarrow\mathcal{A}\sqcap\beta\mathcal{B}\neq 0^{\mathfrak{F}}.$$

The pair $(\alpha;\beta)$ is called *funcoid* when $\mathcal{B}\sqcap\alpha\mathcal{A}\neq 0^{\mathfrak{F}}\Leftrightarrow\mathcal{A}\sqcap\beta\mathcal{B}\neq 0^{\mathfrak{F}}$. So funcoids are a generalization of proximity spaces.

Funcoids consist of two components the first α and the second β. The first component of a funcoid f is denoted as $\langle f\rangle$ and the second component is denoted as $\langle f^{-1}\rangle$. (The similarity of this notation with the notation for the image of a set under a function is not a coincidence, we will see that in the case of principal funcoids (see below) these coincide.)

One of the most important properties of a funcoid is that it is uniquely determined by just one of its components. That is a funcoid f is uniquely determined by the function $\langle f\rangle$. Moreover a funcoid f is uniquely determined by values of $\langle f\rangle$ on principal filters.

Next we will consider some examples of funcoids determined by specified values of the first component on sets.

Funcoids as a generalization of pretopological spaces: Let α be a pretopological space that is a map $\alpha\in\mathfrak{F}^{\mho}$ for some set $\mho$. Then we define $\alpha'\,X\overset{\text{def}}{=}\bigsqcup\,\{\alpha x\mid x\in X\}$ for every set $X\in\mathscr{P}\mho$. We will prove that there exists a unique funcoid f such that $\alpha'=\langle f\rangle|_{\mathfrak{P}}\circ\uparrow$ where $\mathfrak{P}$ is the set of principal filters on $\mho$. So funcoids are a generalization of pretopological spaces. Funcoids are also a generalization of preclosure operators: For every preclosure operator p on a set $\mho$ it exists a unique funcoid f such that $\langle f\rangle|_{\mathfrak{P}}\circ\uparrow=\uparrow\circ p$.

For every binary relation p on a set $\mho$ there exists unique funcoid f such that

$$\forall X\in\mathscr{P}\mho\colon\langle f\rangle\uparrow X=\uparrow\langle p\rangle X$$

(where $\langle p\rangle$ is defined in the introduction), recall that a funcoid is uniquely determined by the values of its first component on sets. I will call such funcoids *principal*. So funcoids are a generalization of binary relations.