δ defined in this way (for a metric space) is an example of proximity as defined below.

Definition 5.29. A *proximity space* is a set $(U;\delta)$ conforming to the following axioms (for every $A, B, C \in \mathscr{P}U$):

1. $A \cap B \neq \emptyset \Rightarrow A\,\delta\,B$;
2. if $A\,\delta\,B$ then $A \neq \emptyset$ and $B \neq \emptyset$;
3. $A\,\delta\,B \Rightarrow B\,\delta\,A$ (*symmetry*);
4. $(A \cup B)\,\delta\,C \Leftrightarrow A\,\delta\,C \vee B\,\delta\,C$;
5. $C\,\delta\,(A \cup B) \Leftrightarrow C\,\delta\,A \vee C\,\delta\,B$;
6. $A\,\bar{\delta}\,B$ implies existence of $P, Q \in \mathscr{P}U$ with $A\,\bar{\delta}\,P$, $B\,\bar{\delta}\,Q$ and $P \cup Q = U$.

Exercise 5.6. Show that proximity generated by a metric space is really a proximity (conforms to the above axioms).

Definition 5.30. *Quasi-proximity* is defined as the above but without the symmetry axiom.

Definition 5.31. Closure is generated by a proximity by the following formula:

$$\mathrm{cl}(A) = \{a \in U \mid \{a\}\,\delta\,A\}.$$

Proposition 5.32. Every closure generated by a proximity is a Kuratowski closure.

Proof. First prove it is a preclosure. $\mathrm{cl}(\emptyset) = \emptyset$ is obvious. $\mathrm{cl}(A) \supseteq A$ is obvious. $\mathrm{cl}(A \cup B) = \{a \in U \mid \{a\}\,\delta\,A \cup B\} = \{a \in U \mid \{a\}\,\delta\,A \vee \{a\}\,\delta\,B\} = \{a \in U \mid \{a\}\,\delta\,A\} \cup \{a \in U \mid \{a\}\,\delta\,B\} = \mathrm{cl}(A) \cup \mathrm{cl}(B)$.

It is remained to prove that cl is idempotent, that is $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$. It is enough to show $\mathrm{cl}(\mathrm{cl}(A)) \subseteq \mathrm{cl}(A)$, that is if $x \notin \mathrm{cl}(A)$ then $x \notin \mathrm{cl}(\mathrm{cl}(A))$.

If $x \notin \mathrm{cl}(A)$ then $\{x\}\,\bar{\delta}\,A$. So there are $P, Q \in \mathscr{P}U$ such that $\{x\}\,\bar{\delta}\,P$, $A\,\bar{\delta}\,Q$, $P \cup Q = U$. Then $U \setminus Q \subseteq P$, so $\{x\}\,\bar{\delta}\,U \setminus Q$ and hence $x \in Q$. Hence $U \setminus \mathrm{cl}(A) \subseteq Q$, and so $\mathrm{cl}(A) \subseteq U \setminus Q \subseteq P$. Consequently $\{x\}\,\bar{\delta}\,\mathrm{cl}(A)$ and hence $x \notin \mathrm{cl}(\mathrm{cl}(A))$. □