2. Union of a finite number of closed sets is a closed set.
3. $\emptyset$ is a closed set.

**Exercise 5.4.** Show that the definitions using open and closed sets are equivalent.

## 5.3.1 Relationships between pretopologies and topologies

### 5.3.1.1 Topological space induced by preclosure space

Having a preclosure space $(U; \mathrm{cl})$ we define a topological space whose closed sets are such sets $A \in \mathscr{P}U$ that $\mathrm{cl}(A) = A$.

**Proposition 5.19.** This really defines a topology.

**Proof.** Let $S$ be a set of closed sets. First, we need to prove that $\bigcap S$ is a closed set. We have $\mathrm{cl}(\bigcap S) \subseteq A$ for every $A \in S$. Thus $\mathrm{cl}(\bigcap S) \subseteq \bigcap S$ and consequently $\mathrm{cl}(\bigcap S) = \bigcap S$. So $\bigcap S$ is a closed set.

Let now $A_0, ..., A_n$ be closed sets, then

$$\mathrm{cl}(A_0 \cup ... \cup A_n) = \mathrm{cl}(A_0) \cup ... \cup \mathrm{cl}(A_n) = A_0 \cup ... \cup A_n$$

that is $A_0 \cup ... \cup A_n$ is a closed set.

That $\emptyset$ is a closed set is obvious. □

Having a pretopological space $(U; \Delta)$ we define a topological space whose open sets are

$$\{X \in \mathscr{P}U \mid \forall x \in X: X \in \Delta(x)\}.$$

**Proposition 5.20.** This really defines a topology.

**Proof.** Let set $S \subseteq \{X \in \mathscr{P}U \mid \forall x \in X: X \in \Delta(x)\}$. Then $\forall X \in S \forall x \in X: X \in \Delta(x)$. Thus

$$\forall x \in \bigcup S \exists X \in S: X \in \Delta(x)$$

and so $\forall x \in \bigcup S: \bigcup S \in \Delta(x)$. So $\bigcup S$ is an open set.

Let now $A_0, ..., A_n \in \{X \in \mathscr{P}U \mid \forall x \in X: X \in \Delta(x)\}$ for $n \in \mathbb{N}$. Then $\forall x \in A_i: A_i \in \Delta(x)$ and so

$$\forall x \in A_0 \cap ... \cap A_n: A_i \in \Delta(x);$$

thus $\forall x \in A_0 \cap ... \cap A_n: A_0 \cap ... \cap A_n \in \Delta(x)$. So $A_0 \cap ... \cap A_n \in \{X \in \mathscr{P}U \mid \forall x \in X: X \in \Delta(x)\}$.

That $U$ is an open set is obvious. □

**Proposition 5.21.** Topology $\tau$ defined by a pretopology and topology $\rho$ defined by the corresponding preclosure, are the same.

**Proof.** Let $A \in \mathscr{P}U$.

$A$ is $\rho$-closed$\Leftrightarrow \mathrm{cl}(A) = A \Leftrightarrow \mathrm{cl}(A) \subseteq A \Leftrightarrow \forall x \in U: (A \in \partial\Delta(x) \Rightarrow x \in A)$;

$A$ is $\tau$-open$\Leftrightarrow \forall x \in A: A \in \Delta(x) \Leftrightarrow \forall x \in U: (x \in A \Rightarrow A \in \Delta(x)) \Leftrightarrow \forall x \in U: (x \notin U \setminus A \Rightarrow U \setminus A \notin \partial\Delta(x))$.

So $\rho$-closed and $\tau$-open are negations of each other. It follows $\rho = \tau$. □

### 5.3.1.2 Preclosure space induced by topological space

We define a preclosure and a pretopology induced by a topology and then show these two are equivalent.

Having a topological space we define a preclosure space by the formula

$$\mathrm{cl}(A) = \bigcap \{X \in \mathscr{P}U \mid X \text{ is a closed set}, X \supseteq A\}.$$

**Proposition 5.22.** It is really a preclosure.