Proposition 5.7. $\operatorname{cl}(A) \supseteq A$.

Proof. It follows from $d(a, a) = 0 < \varepsilon$. □

Exercise 5.2. Prove $\operatorname{cl}(A \cup B) = \operatorname{cl}(A) \cup \operatorname{cl}(B)$ for every subsets A and B of a metric space.

5.1.2 Continuity

Definition 5.8. A function f from a metric space $\mathfrak{A}$ to a metric space $\mathfrak{B}$ is called *continuous* at point $a \in \mathfrak{A}$ when

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in \mathfrak{A}: (d(a, x) < \delta \Rightarrow d(f(a), f(x)) < \varepsilon).$$

Definition 5.9. A function f is called *continuous* when it is continuous at every point of its domain.

5.2 Pretopological spaces

Pretopological space can be defined in two equivalent ways: a *neighborhood system* or a *preclosure operator*. To be more clear I will call *pretopological space* only the first (neighborhood system) and the second call a *preclosure space*.

Definition 5.10. *Pretopological space* is a set U together with a filter $\Delta(x)$ on U for every $x \in U$, such that $\uparrow^U \{x\} \sqsubseteq \Delta(x)$. $\Delta(x)$ is called a *pretopology* on U.

Definition 5.11. *Preclosure* on a set U is an unary operation cl on $\mathscr{P}U$ such that for every A, $B \in \mathscr{P}U$:

1. $\operatorname{cl}(\emptyset) = \emptyset$;
2. $\operatorname{cl}(A) \supseteq A$;
3. $\operatorname{cl}(A \cup B) = \operatorname{cl}(A) \cup \operatorname{cl}(B)$.

I call a preclosure together with a set U as *preclosure space*.

Theorem 5.12. Small pretopological spaces and small preclosure spaces bijectively correspond to each other by the formulas:

$$\operatorname{cl}(A) = \{x \in U \mid A \in \partial \Delta(x)\}; \tag{5.1}$$

$$\Delta(x) = \{A \in \mathscr{P}U \mid x \notin \operatorname{cl}(U \setminus A)\}. \tag{5.2}$$

Proof. First let's prove that cl defined by formula (5.1) is really a preclosure.

$\operatorname{cl}(\emptyset) = \emptyset$ is obvious. If $x \in A$ then $A \in \partial \Delta(x)$ and so $\operatorname{cl}(A) \supseteq A$. $\operatorname{cl}(A \cup B) = \{x \in U \mid A \cup B \in \partial \Delta(x)\} = \{x \in U \mid A \in \partial \Delta(x) \vee B \in \partial \Delta(x)\} = \operatorname{cl}(A) \cup \operatorname{cl}(B)$. So, it is really a preclosure.

Next let's prove that Δ defined by formula (5.2) is a pretopology. That $\Delta(x)$ is an upper set is obvious. Let $A, B \in \Delta(x)$. Then $x \notin \operatorname{cl}(U \setminus A) \wedge x \notin \operatorname{cl}(U \setminus B)$; $x \notin \operatorname{cl}(U \setminus A) \cup \operatorname{cl}(U \setminus B) = \operatorname{cl}((U \setminus A) \cup (U \setminus B)) = \operatorname{cl}(U \setminus (A \cap B))$; $A \cap B \in \Delta(x)$. We have proved that $\Delta(x)$ is a filter.

Let's prove $\uparrow^U \{x\} \sqsubseteq \Delta(x)$. If $A \in \Delta(x)$ then $x \notin \operatorname{cl}(U \setminus A)$ and consequently $x \notin U \setminus A$; $x \in A$; $A \in \uparrow^U \{x\}$. So $\uparrow^U \{x\} \sqsubseteq \Delta(x)$ and thus Δ is a pretopology.

It is left to prove that the functions defined by the above formulas are mutually inverse.

Let cl_0 be a preclosure, let Δ is the pretopology induced by cl_0 by the formula (5.2), let cl_1 is the preclosure induced by Δ by the formula (5.1). Let's prove $\operatorname{cl}_1 = \operatorname{cl}_0$. Really, $x \in \operatorname{cl}_1(A) \Leftrightarrow \Delta(x) \not\asymp \uparrow^U A \Leftrightarrow \forall X \in \Delta(x): X \cap A \neq \emptyset \Leftrightarrow \forall X \in \mathscr{P}U: (x \notin \operatorname{cl}_0(U \setminus X) \Rightarrow X \cap A \neq \emptyset) \Leftrightarrow \forall X' \in \mathscr{P}U: (x \notin \operatorname{cl}_0(X') \Rightarrow A \setminus X' \neq \emptyset) \Leftrightarrow \forall X' \in \mathscr{P}U: (A \setminus X' = \emptyset \Rightarrow x \in \operatorname{cl}_0(X')) \Leftrightarrow \forall X' \in \mathscr{P}U: (A \subseteq X' \Rightarrow x \in \operatorname{cl}_0(X')) \Leftrightarrow x \in \operatorname{cl}_0(A)$. So $\operatorname{cl}_1(A) = \operatorname{cl}_0(A)$.