Remark 4.259. Consequently $[S]$ is atomistic, completely distributive and isomorphic to a power set algebra (see [39]).

Proof. Completeness of $[S]$ is obvious. Let $A \in [S]$. Then there exists $X \in \mathscr{P}S$ such that $A = \bigsqcup^{\mathfrak{F}} X$. Let $B = \bigsqcup^{\mathfrak{F}} (S \setminus X)$. Then $B \in [S]$ and $A \sqcap^{\mathfrak{F}} B = 0^{\mathfrak{F}}$. $A \sqcup^{\mathfrak{F}} B = \bigsqcup^{\mathfrak{F}} S$ is the greatest element of $[S]$. So we have proved that $[S]$ is a boolean lattice.

Now let prove that $[S]$ is atomic with the set of atoms being S. Let $z \in S$ and $A \in [S]$. If $A \neq z$ then either $A = 0^{\mathfrak{F}}$ or $x \in X$ where $A = \bigsqcup^{\mathfrak{F}} X$, $X \in \mathscr{P}S$ and $x \neq z$. Because S is a partition, $\bigsqcup^{\mathfrak{F}} (X \setminus \{z\}) \sqcap^{\mathfrak{F}} z = 0^{\mathfrak{F}}$ and $\bigsqcup^{\mathfrak{F}} (X \setminus \{z\}) \neq 0^{\mathfrak{F}}$. So $A = \bigsqcup^{\mathfrak{F}} X = \bigsqcup^{\mathfrak{F}} (X \setminus \{z\}) \sqcup^{\mathfrak{F}} z \not\sqsubseteq z$.

Finally we will prove that elements of $[S] \setminus S$ are not atoms. Let $A \in [S] \setminus S$ and $A \neq 0$. Then $A \sqsupseteq x \sqcup^{\mathfrak{F}} y$ where $x, y \in S$ and $x \neq y$. If A is an atom then $A = x = y$ what is impossible. □

Proposition 4.260. The conjecture about the value of $[S]$ is equivalent to closedness of $\left\{\bigsqcup^{\mathfrak{F}} X \mid X \in \mathscr{P}S\right\}$ under arbitrary meets and joins.

Proof. If $\left\{\bigsqcup^{\mathfrak{F}} X \mid X \in \mathscr{P}S\right\} = [S]$ then trivially $\left\{\bigsqcup^{\mathfrak{F}} X \mid X \in \mathscr{P}S\right\}$ is closed under arbitrary meets and joins.

If $\left\{\bigsqcup^{\mathfrak{F}} X \mid X \in \mathscr{P}S\right\}$ is closed under arbitrary meets and joins, then it is the complete lattice generated by the set S because it cannot be smaller than the set of all suprema of subsets of S. □

That $\left\{\bigsqcup^{\mathfrak{F}} X \mid X \in \mathscr{P}S\right\}$ is closed under arbitrary joins is trivial. I have not succeeded to prove that it is closed under arbitrary meets, but have proved a weaker statement that is is closed under finite meets:

Proposition 4.261. $\left\{\bigsqcup^{\mathfrak{F}} X \mid X \in \mathscr{P}S\right\}$ is closed under finite meets.

Proof. Let $R = \left\{\bigsqcup^{\mathfrak{F}} X \mid X \in \mathscr{P}S\right\}$. Then for every $X, Y \in \mathscr{P}S$

$$\begin{aligned} \bigsqcup^{\mathfrak{F}} X \sqcap^{\mathfrak{F}} \bigsqcup^{\mathfrak{F}} Y &= \\ \bigsqcup^{\mathfrak{F}} ((X \cap Y) \cup (X \setminus Y)) \sqcap^{\mathfrak{F}} \bigsqcup^{\mathfrak{F}} Y &= \\ \left(\bigsqcup^{\mathfrak{F}} (X \cap Y) \sqcup^{\mathfrak{F}} \bigsqcup^{\mathfrak{F}} (X \setminus Y)\right) \sqcap^{\mathfrak{F}} \bigsqcup^{\mathfrak{F}} Y &= \\ \left(\bigsqcup^{\mathfrak{F}} (X \cap Y) \sqcap^{\mathfrak{F}} \bigsqcup^{\mathfrak{F}} Y\right) \sqcup^{\mathfrak{F}} \left(\bigsqcup^{\mathfrak{F}} (X \setminus Y) \sqcap^{\mathfrak{F}} \bigsqcup^{\mathfrak{F}} Y\right) &= \\ \left(\bigsqcup^{\mathfrak{F}} (X \cap Y) \sqcap^{\mathfrak{F}} \bigsqcup^{\mathfrak{F}} Y\right) \sqcup^{\mathfrak{F}} 0^{\mathfrak{F}} &= \\ \bigsqcup^{\mathfrak{F}} (X \cap Y) \sqcap^{\mathfrak{F}} \bigsqcup^{\mathfrak{F}} Y. \end{aligned}$$

Applying the formula $\bigsqcup^{\mathfrak{F}} X \sqcap^{\mathfrak{F}} \bigsqcup^{\mathfrak{F}} Y = \bigsqcup^{\mathfrak{F}} (X \cap Y) \sqcap^{\mathfrak{F}} \bigsqcup^{\mathfrak{F}} Y$ twice we get

$$\begin{aligned} \bigsqcup^{\mathfrak{F}} X \sqcap^{\mathfrak{F}} \bigsqcup^{\mathfrak{F}} Y &= \\ \bigsqcup^{\mathfrak{F}} (X \cap Y) \sqcap^{\mathfrak{F}} \bigsqcup^{\mathfrak{F}} (Y \cap (X \cap Y)) &= \\ \bigsqcup^{\mathfrak{F}} (X \cap Y) \sqcap^{\mathfrak{F}} \bigsqcup^{\mathfrak{F}} (X \cap Y) &= \\ \bigsqcup^{\mathfrak{F}} (X \cap Y). \end{aligned}$$

But for any $A, B \in R$ there exist $X, Y \in \mathscr{P}S$ such that $A = \bigsqcup^{\mathfrak{F}} X$, $B = \bigsqcup^{\mathfrak{F}} Y$. So $A \sqcap^{\mathfrak{F}} B = \bigsqcup^{\mathfrak{F}} X \sqcap^{\mathfrak{F}} \bigsqcup^{\mathfrak{F}} Y = \bigsqcup^{\mathfrak{F}} (X \cap Y) \in R$. □