Suppose $Z \neq Z' \in \mathscr{P}\mathbb{N}$. Without loss of generality we may assume that some $b \in Z$ but $b \notin Z'$. Then $M(Z) \in \mathcal{F}_b$ and $\mathbb{N} \setminus M(Z') \in \mathcal{F}_b$. If $M(Z) = M(Z')$ then $\mathcal{F}_b = 0^{\mathfrak{F}}$ what contradicts to the above.

So M is an injective function from $\mathscr{P}\mathbb{R}$ to $\mathscr{P}\mathbb{N}$ what is impossible due cardinality issues. □

Lemma 4.253. (by Niels Diepeveen, with help of Karl Kronenfeld) Let K be a collection of free ultrafilters. We have $\bigsqcup K = \Omega$ iff $\exists \mathcal{G} \in K: A \in \mathcal{G}$ for every infinite set A.

Proof.

⇒. Suppose $\bigsqcup K = \Omega$ and let A be a set such that $\nexists \mathcal{G} \in K: A \in \mathcal{G}$. Let's prove A is finite.
Really, $\forall \mathcal{G} \in K: \mathfrak{U} \setminus A \in \mathcal{G}$; $\mathfrak{U} \setminus A \in \Omega$; A is finite.

⇐. Let $\exists \mathcal{G} \in K: A \in \mathcal{G}$. Suppose A is a set in $\bigsqcup K$.
To finish the proof it's enough to show that $\mathfrak{U} \setminus A$ is finite.
Suppose $\mathfrak{U} \setminus A$ is infinite. Then $\exists \mathcal{G} \in K: \mathfrak{U} \setminus A \in \mathcal{G}$; $\exists \mathcal{G} \in K: A \notin \mathcal{G}$; $A \notin \bigsqcup K$, contradiction. □

Lemma 4.254. (by Niels Diepeveen) If K is a non-empty set of ultrafilters such that $\bigsqcup K = \Omega$, then for every $\mathcal{G} \in K$ we have $\bigsqcup (K \setminus \{\mathcal{G}\}) = \Omega$.

Proof. $\exists \mathcal{F} \in K: A \in \mathcal{F}$ for every infinite set A.
The set A can be partitioned into two infinite sets A_1, A_2.
Take $\mathcal{F}_1, \mathcal{F}_2 \in K$ such that $A_1 \in \mathcal{F}_1$, $A_2 \in \mathcal{F}_2$.
$\mathcal{F}_1 \neq \mathcal{F}_2$ because otherwise A_1 and A_2 are not disjoint.
Obviously $A \in \mathcal{F}_1$ and $A \in \mathcal{F}_2$.
So there exist two different $\mathcal{F} \in K$ such that $A \in \mathcal{F}$. Consequently
$\exists \mathcal{F} \in K \setminus \{\mathcal{G}\}: A \in \mathcal{F}$ that is $\bigsqcup (K \setminus \{\mathcal{G}\}) = \Omega$. □

Example 4.255. There exists a filter on a set which cannot be weakly partitioned into ultrafilters.

Proof. Consider cofinite filter Ω on any infinite set.
Suppose K is its weak partition into ultrafilters. Then $x \asymp \bigsqcup (K \setminus \{x\})$ for some ultrafilter $x \in K$.
We have $\bigsqcup (K \setminus \{x\}) \sqsubset \bigsqcup K$ (otherwise $x \sqsubseteq \bigsqcup (K \setminus \{x\})$) what is impossible due the last lemma. □

Corollary 4.256. There exists a filter on a set which cannot be strongly partitioned into ultrafilters.

4.6 Open problems about filters

In this section, I will formulate some conjectures about lattices of filters on a set. If a conjecture comes true, it may be generalized for more general lattices (such as, for example, lattices of filters on arbitrary lattices). I deem that the main challenge is to prove the special case about lattices of filters on a set, and generalizing the conjectures is expected to be an easy task.

4.6.1 Partitioning

Consider the complete lattice $[S]$ generated by the set S where S is a strong partition of some element a.

Conjecture 4.257. $[S] = \left\{ \bigsqcup^{\mathfrak{F}} X \mid X \in \mathscr{P}S \right\}$, where $[S]$ is the complete lattice generated by a strong partition S of filter on a set.

Consider also the similar conjecture with weak partition instead strong partition.

Proposition 4.258. Provided that the last conjecture is true, we have that $[S]$ is a complete atomic boolean lattice with the set of its atoms being S.