Choose for every $i \in \mathbb{N}$ some $z_i \in X \cap A_i$. Then $\{z_0, z_1, \ldots\}$ is an infinite subset of X (take into account that $z_i \neq z_j$ for $i \neq j$). Let $Y = X \setminus \{z_0, z_1, \ldots\}$. Then $\uparrow Y \sqcap^{\mathfrak{F}} \Omega \sqsupseteq \uparrow A_i \sqcap^{\mathfrak{F}} \Omega$ because $A_i \setminus Y = A_i \setminus (X \setminus \{z_i\}) = (A_i \setminus X) \cup \{z_i\}$ which is finite because $A_i \setminus X$ is finite. Thus $[Y]$ is an upper bound for $\{[A_0], [A_1], \ldots\}$.

Suppose $\uparrow Y \sqcap^{\mathfrak{F}} \Omega = \uparrow X \sqcap^{\mathfrak{F}} \Omega$. Then $Y \setminus X$ is finite what is not true. So $\uparrow Y \sqcap^{\mathfrak{F}} \Omega \sqsubset \uparrow X \sqcap^{\mathfrak{F}} \Omega$ that is $[Y]$ is below $[X]$. □

4.5.1 Weak and Strong Partition

Definition 4.249. A family S of subsets of a countable set is *independent* iff the intersection of any finitely many members of S and the complements of any other finitely many members of S is infinite.

Lemma 4.250. The "infinite" at the end of the definition could be equivalently replaced with "non-empty" if we assume that S is infinite.

Proof. Suppose that some sets from the above definition has a finite intersection J of cardinality n. Then (thanks S is infinite) get one more set $X \in S$ and we have $J \cap X \neq \emptyset$ and $J \cap (\mathbb{N} \setminus X) \neq \emptyset$. So $\operatorname{card}(J \cap X) < n$. Repeating this, we prove that for some finite family of sets we have empty intersection what is a contradiction. □

Lemma 4.251. There exists an independent family on $\mathbb{N}$ of cardinality $\mathfrak{c}$.

Proof. Let C be the set of finite subsets of $\mathbb{Q}$. Since $\operatorname{card} C = \operatorname{card} \mathbb{N}$, it suffices to find $\mathfrak{c}$ independent subsets of C. For each $r \in \mathbb{R}$ let

$$E_r = \{F \in C \mid \operatorname{card}(F \cap (-\infty; r)) \text{ is even}\}.$$

All E_{r_1} and E_{r_2} are distinct for distinct $r_1, r_2 \in \mathbb{R}$ since we may consider $F = \{r'\} \in C$ where a rational number r' is between r_1 and r_2 and thus F is a member of exactly one of the sets E_{r_1} and E_{r_2}. Thus $\operatorname{card}\{E_r \mid r \in \mathbb{R}\} = \mathfrak{c}$.

We will show that $\{E_r \mid r \in \mathbb{R}\}$ is independent. Let $r_1, \ldots, r_k, s_1, \ldots, s_k$ be distinct reals. It is enough to show that these have a nonempty intersection, that is existence of some F such that F belongs to all the E_r and none of E_s.

But this can be easily accomplished taking F having zero or one element in each of intervals to which $r_1, \ldots, r_k, s_1, \ldots, s_k$ split the real line. □

Example 4.252. There exists a weak partition of a filter on a set which is not a strong partition.

Proof. (suggested by Andreas Blass) Let $\{X_r \mid r \in \mathbb{R}\}$ be an independent family of subsets of $\mathbb{N}$. We can assume $a \neq b \Rightarrow X_a \neq X_b$ due the above lemma.

Let $\mathcal{F}_a$ be a filter generated by X_a and the complements $\mathbb{N} \setminus X_b$ for all $b \in \mathbb{R}$, $b \neq a$. Independence implies that $\mathcal{F}_a \neq 0^{\mathfrak{F}}$ (by properties of filter bases).

Let $S = \{\mathcal{F}_r \mid r \in \mathbb{R}\}$. We will prove that S is a weak partition but not a strong partition.

Let $a \in \mathbb{R}$. Then $X_a \in \mathcal{F}_a$ while $\forall b \in \mathbb{R} \setminus \{a\}$: $\mathbb{N} \setminus X_a \in \mathcal{F}_b$ and therefore $\mathbb{N} \setminus X_a \in \bigsqcup^{\mathfrak{F}} \{\mathcal{F}_b \mid \mathbb{R} \ni b \neq a\}$. Therefore $\mathcal{F}_a \sqcap^{\mathfrak{F}} \bigsqcup^{\mathfrak{F}} \{\mathcal{F}_b \mid \mathbb{R} \ni b \neq a\} = 0^{\mathfrak{F}}$. Thus S is a weak partition.

Suppose S is a strong partition. Then for each set $Z \in \mathscr{P}\mathbb{R}$

$$\bigsqcup^{\mathfrak{F}} \{\mathcal{F}_b \mid b \in Z\} \sqcap^{\mathfrak{F}} \bigsqcup^{\mathfrak{F}} \{\mathcal{F}_b \mid b \in \mathbb{R} \setminus Z\} = 0^{\mathfrak{F}}$$

what is equivalent to existence of $M(Z) \in \mathscr{P}\mathbb{N}$ such that

$$M(Z) \in \bigsqcup^{\mathfrak{F}} \{\mathcal{F}_b \mid b \in Z\} \quad \text{and} \quad \mathbb{N} \setminus M(Z) \in \bigsqcup^{\mathfrak{F}} \{\mathcal{F}_b \mid b \in \mathbb{R} \setminus Z\}$$

that is

$$\forall b \in Z: M(Z) \in \mathcal{F}_b \quad \text{and} \quad \forall b \in \mathbb{R} \setminus Z: \mathbb{N} \setminus M(Z) \in \mathcal{F}_b.$$