

### 4.4.2 Number of Filters on a Set

**Definition 4.238.** A collection  $Y$  of sets has *finite intersection property* iff intersection of any finite subcollection of  $Y$  is non-empty.

The following was borrowed from [7]. Thanks to Andreas Blass for email support about his proof.

**Lemma 4.239.** (by Hausdorff) For an infinite set  $X$  there is a family  $\mathcal{F}$  of  $2^{\text{card } X}$  many subsets of  $X$  such that given any disjoint finite subfamilies  $\mathcal{A}, \mathcal{B}$ , the intersection of sets in  $\mathcal{A}$  and complements of sets in  $\mathcal{B}$  is nonempty.

**Proof.** Let

$$X' = \{(P; Q) \mid P \in \mathcal{P}X \text{ is finite, } Q \in \mathcal{P}\mathcal{P}X\}.$$

It's easy to show that  $\text{card } X' = \text{card } X$ . So it is enough to show this for  $X'$  instead of  $X$ . Let

$$\mathcal{F} = \{\{(P; Q) \in X' \mid Y \cap P \in Q\} \mid Y \in \mathcal{P}X\}.$$

To finish the proof we show that for every disjoint finite  $Y_+ \in \mathcal{P}\mathcal{P}X$  and finite  $Y_- \in \mathcal{P}\mathcal{P}X$  there exist  $(P; Q) \in X'$  such that

$$\forall Y \in Y_+: (P; Q) \in \{(P; Q) \in X' \mid Y \cap P \in Q\} \quad \text{and} \quad \forall Y \in Y_-: (P; Q) \notin \{(P; Q) \in X' \mid Y \cap P \in Q\}$$

what is equivalent to existence  $(P; Q) \in X'$  such that

$$\forall Y \in Y_+: Y \cap P \in Q \quad \text{and} \quad \forall Y \in Y_-: Y \cap P \notin Q.$$

For existence of this  $(P; Q)$ , it is enough existence of  $P$  such that intersections  $Y \cap P$  are different for different  $Y \in Y_+ \cup Y_-$ .

Really, for each pair of distinct  $Y_0, Y_1 \in Y_+ \cup Y_-$  choose a point which lies in one of the sets  $Y_0, Y_1$  and not in an other, and call the set of such points  $P$ . Then  $Y \cap P$  are different for different  $Y \in Y_+ \cup Y_-$ .  $\square$

**Corollary 4.240.** For an infinite set  $X$  there is a family  $\mathcal{F}$  of  $2^{\text{card } X}$  many subsets of  $X$  such that for arbitrary disjoint subfamilies  $\mathcal{A}$  and  $\mathcal{B}$  the set  $\mathcal{A} \cup \{X \setminus A \mid A \in \mathcal{B}\}$  has finite intersection property.

**Theorem 4.241.** Let  $X$  be a set. The number of ultrafilters on  $X$  is  $2^{2^{\text{card } X}}$  if  $X$  is infinite and  $\text{card } X$  if  $X$  is finite.

**Proof.** The finite case follows from the fact that every ultrafilter on a finite set is trivial. Let  $X$  be infinite. From the lemma, there exists a family  $\mathcal{F}$  of  $2^{\text{card } X}$  many subsets of  $X$  such that for every  $\mathcal{G} \in \mathcal{P}\mathcal{F}$  we have  $\Phi(\mathcal{F}; \mathcal{G}) = \prod \langle \uparrow \rangle \mathcal{G} \prod \prod \langle \uparrow \rangle \{X \setminus A \mid A \in \mathcal{F} \setminus \mathcal{G}\} \neq 0^{\mathfrak{S}(X)}$ .

This filter contains all sets from  $\mathcal{G}$  and does not contain any sets from  $\mathcal{F} \setminus \mathcal{G}$ . So for every suitable pairs  $(\mathcal{F}_0; \mathcal{G}_0)$  and  $(\mathcal{F}_1; \mathcal{G}_1)$  there is  $A \in \Phi(\mathcal{F}_0; \mathcal{G}_0)$  such that  $\bar{A} \in \Phi(\mathcal{F}_1; \mathcal{G}_1)$ . Consequently all filters  $\Phi(\mathcal{F}; \mathcal{G})$  are disjoint. So for every pair  $(\mathcal{F}; \mathcal{G})$  where  $\mathcal{G} \in \mathcal{P}\mathcal{F}$  there exist a distinct ultrafilter under  $\Phi(\mathcal{F}; \mathcal{G})$ , but the number of such pairs  $(\mathcal{F}; \mathcal{G})$  is  $2^{2^{\text{card } X}}$ . Obviously the number of all filters is not above  $2^{2^{\text{card } X}}$ .  $\square$

**Corollary 4.242.** The number of filters on  $\mathfrak{U}$  is  $2^{2^{\text{card } \mathfrak{U}}}$  if  $\mathfrak{U}$  is infinite and  $2^{\text{card } \mathfrak{U}}$  if  $\mathfrak{U}$  is finite.

**Proof.** The finite case is obvious. The infinite case follows from the theorem and the fact that filters are collections of sets and there cannot be more than  $2^{2^{\text{card } \mathfrak{U}}}$  collections of sets on  $\mathfrak{U}$ .  $\square$

## 4.5 Some Counter-Examples

**Example 4.243.** There exist a bounded distributive lattice which is not lattice with separable center.