4.3.22 Pseudodifference of filters

Proposition 4.167. For a lattice $\mathfrak{F}$ of filters over a boolean lattice and a, $b \in \mathfrak{F}$ the following expressions are always equal:

1. $a \setminus^* b = \bigsqcap \{ z \in \mathfrak{F} \mid a \sqsubseteq b \sqcup z \}$ (quasidifference of a and b);
2. $a \# b = \bigsqcup \{ z \in \mathfrak{F} \mid z \sqsubseteq a \wedge z \sqcap b = 0 \}$ (second quasidifference of a and b);
3. $\bigsqcup (\operatorname{atoms} a \setminus \operatorname{atoms} b)$.

Proof. Theorem 3.43, taking into account corollary 4.115 theorem 4.137. □

4.4 Filters on a Set

In this section we will consider filters on the poset $\mathfrak{Z} = \mathscr{P}\mathfrak{U}$ (where $\mathfrak{U}$ is some fixed set) with the order $A \sqsubseteq B \Leftrightarrow A \subseteq B$ (for $A, B \in \mathscr{P}\mathfrak{U}$).

In fact, it is a complete atomistic boolean lattice with $\bigsqcap S = \bigcap S$, $\bigsqcup S = \bigcup S$, $\overline{A} = \mathfrak{U} \setminus A$ for every $S \in \mathscr{P}\mathscr{P}\mathfrak{U}$ and $A \in \mathscr{P}\mathfrak{U}$, atoms being one-element sets.

Definition 4.168. I will call a filter on the lattice of all subsets of a given set $\mathfrak{U}$ as a *filter on set*.

Definition 4.169. I will denote the set on which a filter $\mathcal{F}$ is defined as $\operatorname{Base}(\mathcal{F})$.

Obvious 4.170. $\operatorname{Base}(\mathcal{F}) = \bigcup \mathcal{F}$.

Definition 4.171. I will call the primary filtrator for $\mathfrak{Z} = \mathscr{P}\mathfrak{U}$ (with order on $\mathfrak{Z}$ defined as $A \sqsubseteq B \Leftrightarrow A \subseteq B$) for some set $\mathfrak{U}$ as *powerset filtrator*.

Proposition 4.172. The following are equivalent for a non-empty set $F \in \mathscr{P}\mathscr{P}\mathfrak{U}$:

1. F is a filter.
2. $\forall X, Y \in F: X \cap Y \in F$ and F is an upper set.
3. $\forall X, Y \in \mathscr{P}\mathfrak{U}: (X, Y \in F \Leftrightarrow X \cap Y \in F)$.

Proof. By theorem 4.82. □

Obvious 4.173. The minimal filter on $\mathscr{P}\mathfrak{U}$ is $\mathscr{P}\mathfrak{U}$.

Obvious 4.174. The maximal filter on $\mathscr{P}\mathfrak{U}$ is $\{\mathfrak{U}\}$.

I will denote $\uparrow A = \uparrow^{\mathfrak{U}} A = \uparrow^{\mathscr{P}\mathfrak{U}} A$. (The distinction between conflicting notations $\uparrow^{\mathfrak{U}} A$ and $\uparrow^{\mathscr{P}\mathfrak{U}} A$ will be clear from the context.)

Proposition 4.175. The powerset filtrator is both up-aligned and down-aligned.

Proof. By theorem 4.98. □

Proposition 4.176. Every powerset filtrator is filtered.

Proof. By corollary 4.95. □

Proposition 4.177. Every powerset filtrator is with join-closed core.

Proof. By corollary 4.96. □

Proposition 4.178. Every powerset filtrator is with finitely meet-closed core.

Proof. By proposition 4.97. □