Proof. $\exists A\in\mathcal{A}\colon X\sqcap^{\mathfrak{Z}}A=Y\sqcap^{\mathfrak{Z}}A\Leftrightarrow\exists A\in\mathcal{A}\colon\uparrow X\sqcap^{\mathfrak{F}}\uparrow A=\uparrow Y\sqcap^{\mathfrak{F}}\uparrow A\Rightarrow\exists A\in\mathcal{A}\colon\uparrow X\sqcap^{\mathfrak{F}}\uparrow A\sqcap^{\mathfrak{F}}\mathcal{A}=\uparrow Y\sqcap^{\mathfrak{F}}\uparrow A\sqcap^{\mathfrak{F}}\mathcal{A}\Leftrightarrow\exists A\in\mathcal{A}\colon\uparrow X\sqcap^{\mathfrak{F}}\mathcal{A}=\uparrow Y\sqcap^{\mathfrak{F}}\mathcal{A}\Leftrightarrow\uparrow X\sqcap^{\mathfrak{F}}\mathcal{A}=\uparrow Y\sqcap^{\mathfrak{F}}\mathcal{A}\Leftrightarrow\uparrow X\sim\uparrow Y\Leftrightarrow X\sim Y$.

On the other hand, $\uparrow X\sqcap^{\mathfrak{F}}\mathcal{A}=\uparrow Y\sqcap^{\mathfrak{F}}\mathcal{A}\Leftrightarrow\{X\sqcap^{\mathfrak{Z}}A_0\mid A_0\in\mathcal{A}\}=\{Y\sqcap^{\mathfrak{Z}}A_1\mid A_1\in\mathcal{A}\}\Rightarrow\exists A_0,A_1\in\mathcal{A}\colon X\sqcap^{\mathfrak{Z}}A_0=Y\sqcap^{\mathfrak{Z}}A_1\Rightarrow\exists A_0,A_1\in\mathcal{A}\colon X\sqcap^{\mathfrak{Z}}A_0\sqcap^{\mathfrak{Z}}A_1=Y\sqcap^{\mathfrak{Z}}A_0\sqcap^{\mathfrak{Z}}A_1\Rightarrow\exists A\in\mathcal{A}\colon X\sqcap^{\mathfrak{Z}}A=Y\sqcap^{\mathfrak{Z}}A$. □

Proposition 4.163. The relation $\sim$ is a congruence[4.1] for each of the following:

1. a meet-semilattice $\mathfrak{A}$;
2. a distributive lattice $\mathfrak{A}$.

Proof. Let $a_0,a_1,b_0,b_1\in\mathfrak{A}$ and $a_0\sim a_1$ and $b_0\sim b_1$.

1. $a_0\sqcap b_0\sim a_1\sqcap b_1$ because $(a_0\sqcap b_0)\sqcap\mathcal{A}=a_0\sqcap(b_0\sqcap\mathcal{A})=a_0\sqcap(b_1\sqcap\mathcal{A})=b_1\sqcap(a_0\sqcap\mathcal{A})=b_1\sqcap(a_1\sqcap\mathcal{A})=(a_1\sqcap b_1)\sqcap\mathcal{A}$.
2. Taking the above into account, we need to prove only $a_0\sqcup b_0\sim a_1\sqcup b_1$. We have

$$(a_0\sqcup b_0)\sqcap\mathcal{A}=(a_0\sqcap\mathcal{A})\sqcup(b_0\sqcap\mathcal{A})=(a_1\sqcap\mathcal{A})\sqcup(b_1\sqcap\mathcal{A})=(a_1\sqcup b_1)\sqcap\mathcal{A}.$$ □

Definition 4.164. We will denote $A/(\sim)=A/((\sim)\cap A\times A)$ for a set A and an equivalence relation $\sim$ on a set $B\supseteq A$. I will call $\sim$ a congruence on A when $(\sim)\cap A\times A$ is a congruence on A.

Theorem 4.165. Let $\mathfrak{F}$ be the set of filters over a boolean lattice $\mathfrak{Z}$ and $\mathcal{A}\in\mathfrak{F}$. Consider the function $\gamma\colon Z(D\mathcal{A})\to\mathfrak{Z}/\sim$ defined by the formula (for every $p\in Z(D\mathcal{A})$)

$$\gamma p=\{X\in\mathfrak{Z}\mid\uparrow X\sqcap^{\mathfrak{F}}\mathcal{A}=p\}.$$

Then:

1. γ is a lattice isomorphism.
2. $\forall Q\in q\colon\gamma^{-1}q=\uparrow Q\sqcap^{\mathfrak{F}}\mathcal{A}$ for every $q\in\mathfrak{Z}/\sim$.

Proof. $\forall p\in Z(D\mathcal{A})\colon\gamma p\neq\emptyset$ because of theorem 4.147. Thus it is easy to see that $\gamma p\in\mathfrak{Z}/\sim$ and that γ is an injection.

Let's prove that γ is a lattice homomorphism:

$\gamma(p_0\sqcap^{\mathfrak{F}}p_1)=\{X\in\mathfrak{Z}\mid\uparrow X\sqcap^{\mathfrak{F}}\mathcal{A}=p_0\sqcap^{\mathfrak{F}}p_1\}$;

$$\begin{aligned}\gamma p_0\sqcap^{\mathfrak{Z}/\sim}\gamma p_1 &=\\ \{X_0\in\mathfrak{Z}\mid\uparrow X_0\sqcap^{\mathfrak{F}}\mathcal{A}=p_0\}\sqcap^{\mathfrak{Z}/\sim}\{X_1\in\mathfrak{Z}\mid\uparrow X_1\sqcap^{\mathfrak{F}}\mathcal{A}=p_1\} &=\\ \{\uparrow X_0\sqcap^{\mathfrak{F}}\uparrow X_1\mid X_0,X_1\in\mathfrak{Z},\uparrow X_0\sqcap^{\mathfrak{F}}\mathcal{A}=p_0\wedge\uparrow X_1\sqcap^{\mathfrak{F}}\mathcal{A}=p_1\} &\subseteq\\ \{X'\in\mathfrak{Z}\mid\uparrow X'\sqcap^{\mathfrak{F}}\mathcal{A}=p_0\sqcap^{\mathfrak{F}}p_1\} &=\\ \gamma(p_0\sqcap^{\mathfrak{F}}p_1). &\end{aligned}$$

Because $\gamma p_0\sqcap^{\mathfrak{Z}/\sim}\gamma p_1$ and $\gamma(p_0\sqcap^{\mathfrak{F}}p_1)$ are equivalence classes, thus follows $\gamma p_0\sqcap^{\mathfrak{Z}/\sim}\gamma p_1=\gamma(p_0\sqcap^{\mathfrak{F}}p_1)$.

To finish the proof it is enough to show that $\forall Q\in q\colon q=\gamma(\uparrow Q\sqcap^{\mathfrak{F}}\mathcal{A})$ for every $q\in\mathfrak{Z}/\sim$. (From this it follows that γ is surjective because q is not empty and thus $\exists Q\in q\colon q=\gamma(\uparrow Q\sqcap^{\mathfrak{F}}\mathcal{A})$.) Really,

$$\gamma(\uparrow Q\sqcap^{\mathfrak{F}}\mathcal{A})=\{X\in\mathfrak{Z}\mid\uparrow X\sqcap^{\mathfrak{F}}\mathcal{A}=\uparrow Q\sqcap^{\mathfrak{F}}\mathcal{A}\}=[Q]=q.$$ □

This isomorphism is useful in both directions to reveal properties of both lattices $Z(D\mathcal{A})$ and $\mathfrak{Z}/\sim$.

Corollary 4.166. If $\mathfrak{Z}$ is a boolean lattice then $\mathfrak{Z}/\sim$ is a boolean lattice.

Proof. Because $Z(D\mathcal{A})$ is a boolean lattice (theorem 2.79). □

4.1. See Wikipedia for a definition of congruence.