Let $A\sqcup^{\mathfrak{P}}B\in\bigcup\langle\partial\rangle S$. Then there exists $Q\in\langle\partial\rangle S$ such that $A\sqcup^{\mathfrak{P}}B\in Q$. Then $A\in Q\vee B\in Q$, consequently $A\in\bigcup\langle\partial\rangle S\vee B\in\bigcup\langle\partial\rangle S$. Let now $A\in\bigcup\langle\partial\rangle S$. Then there exists $Q\in\langle\partial\rangle S$ such as $A\in Q$, consequently $A\sqcup^{\mathfrak{P}}B\in Q$ and $A\sqcup^{\mathfrak{P}}B\in\bigcup\langle\partial\rangle S$. □

4.3.15 More about the Lattice of Filters

Definition 4.133. Atoms of $\mathfrak{F}$ (for any poset $\mathfrak{Z}$) are called *ultrafilters*.

Definition 4.134. Principal ultrafilters are also called *trivial ultrafilters*.

Theorem 4.135. If $\mathfrak{Z}$ is a bounded distributive lattice [TODO: Generalize for meet-semilattices?] with least element then $\mathfrak{F}$ is an atomic lattice.

Proof. Let $\mathcal{F}\in\mathfrak{F}$. Let choose (by Kuratowski's lemma) a maximal chain S from $0^{\mathfrak{F}}$ to $\mathcal{F}$. Let $S'=S\setminus\{0^{\mathfrak{F}}\}$. $a=\bigsqcap^{\mathfrak{F}}S'\neq 0^{\mathfrak{F}}$ by properties of generalized filter bases (the corollary 4.122 which uses the fact that $\mathfrak{Z}$ is a distributive lattice with least element). If $a\notin S$ then the chain S can be extended adding there element a because $0^{\mathfrak{F}}\sqsubset a\sqsubseteq\mathcal{X}$ for any $\mathcal{X}\in S'$ what contradicts to maximality of the chain. So $a\in S$ and consequently $a\in S'$. Obviously a is the minimal element of S'. Consequently (taking into account maximality of the chain) there is no $\mathcal{Y}\in\mathfrak{F}$ such that $0^{\mathfrak{F}}\sqsubset\mathcal{Y}\sqsubset a$. So a is an atomic filter. Obviously $a\sqsubseteq\mathcal{F}$. □

Obvious 4.136. If $\mathfrak{Z}$ is a boolean lattice then $\mathfrak{F}$ is separable.

Theorem 4.137. If $\mathfrak{Z}$ is a boolean lattice then $\mathfrak{F}$ is an atomistic lattice.

Proof. Because (used the theorem 3.20) $\mathfrak{F}$ is atomic (theorem 4.135) and separable. □

Corollary 4.138. If $\mathfrak{Z}$ is a boolean lattice then $\mathfrak{F}$ is atomically separable.

Proof. By theorem 3.19. □

Theorem 4.139. When $\mathfrak{Z}$ is a boolean lattice, the filtrator $(\mathfrak{F};\mathfrak{P})$ is central.

Proof. We can conclude that $\mathfrak{F}$ is atomically separable (the corollary 4.138), with separable core (the theorem 4.112), and with join-closed core (corollary 4.96).

We need to prove $Z(\mathfrak{F})=\mathfrak{P}$.

Let $\mathcal{X}\in Z(\mathfrak{F})$. Then there exists $\mathcal{Y}\in Z(\mathfrak{F})$ such that $\mathcal{X}\sqcap^{\mathfrak{F}}\mathcal{Y}=0^{\mathfrak{F}}$ and $\mathcal{X}\sqcup^{\mathfrak{F}}\mathcal{Y}=1^{\mathfrak{F}}$. Consequently there is $X\in\operatorname{up}\mathcal{X}$ such that $X\sqcap^{\mathfrak{F}}\mathcal{Y}=0^{\mathfrak{F}}$; we also have $X\sqcup^{\mathfrak{F}}\mathcal{Y}=1^{\mathfrak{F}}$. Suppose $X\sqsupset\mathcal{X}$. Then there exists $a\in\operatorname{atoms}^{\mathfrak{F}}X$ such that $a\notin\operatorname{atoms}^{\mathfrak{F}}\mathcal{X}$. We can conclude also $a\notin\operatorname{atoms}^{\mathfrak{F}}\mathcal{Y}$ (otherwise $X\sqcap^{\mathfrak{F}}\mathcal{Y}\neq 0^{\mathfrak{F}}$). Thus $a\notin\operatorname{atoms}^{\mathfrak{F}}(\mathcal{X}\sqcup^{\mathfrak{F}}\mathcal{Y})$ and consequently $\mathcal{X}\sqcup^{\mathfrak{F}}\mathcal{Y}\neq 1^{\mathfrak{F}}$ what is a contradiction. We have $\mathcal{X}=X\in\mathfrak{P}$.

Let now $X\in\mathfrak{P}$. Let $Y=\overline{X}$. We have $X\sqcap^{\mathfrak{P}}Y=0^{\mathfrak{F}}$ and $X\sqcup^{\mathfrak{P}}Y=1^{\mathfrak{F}}$. Thus $X\sqcap^{\mathfrak{F}}Y=\bigsqcap^{\mathfrak{P}}\{X\sqcap^{\mathfrak{P}}Y\}=0^{\mathfrak{F}}$; $X\sqcup^{\mathfrak{F}}Y=X\sqcup^{\mathfrak{P}}Y=1^{\mathfrak{F}}$. We have shown that $X\in Z(\mathfrak{F})$. □

4.3.16 Atomic Filters

Proposition 4.140. If $\mathfrak{Z}$ is a meet-semilattice with least element, then a is an atom of $\mathfrak{P}$ iff $a\in\mathfrak{P}$ and a is an atom of $\mathfrak{F}$.

Proof. It is semifiltered by the corollary 4.95, finitely meet-closed by proposition 4.97. So we can apply the theorem 4.49. □

Proposition 4.141. If $\mathfrak{Z}$ is a meet-semilattice with least element then, $a\in\mathfrak{F}$ is an atom of $\mathfrak{F}$ iff $\operatorname{up}a=\partial a$.

Proof. It is semifiltered by the corollary 4.95, $\mathfrak{F}$ is a meet-semilattice by the corollary 4.107. So we can apply theorem 4.50. □