For boolean lattices free stars bijectively correspond to filters:

Theorem 4.129. If $\mathfrak{Z}$ is a boolean lattice, then for any set $S \in \mathscr{P}\mathfrak{P}$ there exists a filter $\mathcal{A}$ such that $\partial\mathcal{A} = S$ iff S is a free star.

Proof.

⇒. That $0^{\mathfrak{P}} \notin S$ is obvious. For every $A, B \in \mathfrak{A}$

$$\begin{aligned} A \sqcup^{\mathfrak{P}} B \in S &\Leftrightarrow \\ (A \sqcup^{\mathfrak{P}} B) \sqcap^{\mathfrak{F}} \mathcal{A} \neq 0^{\mathfrak{F}} &\Leftrightarrow \\ (A \sqcup^{\mathfrak{F}} B) \sqcap^{\mathfrak{F}} \mathcal{A} \neq 0^{\mathfrak{F}} &\Leftrightarrow \\ (A \sqcap^{\mathfrak{F}} \mathcal{A}) \sqcup^{\mathfrak{F}} (B \sqcap^{\mathfrak{F}} \mathcal{A}) \neq 0^{\mathfrak{F}} &\Leftrightarrow \\ A \sqcap^{\mathfrak{F}} \mathcal{A} \neq 0^{\mathfrak{F}} \vee B \sqcap^{\mathfrak{F}} \mathcal{A} \neq 0^{\mathfrak{F}} &\Leftrightarrow \\ A \in S \vee B \in S & \end{aligned}$$

(taken into account corollary 4.114 and theorem 4.25).

⇐. $0^{\mathfrak{P}} \notin S$ and $\forall A, B \in S: (A \sqcup^{\mathfrak{P}} B \in S \Leftrightarrow A \in S \vee B \in S)$. Let $T = \{\overline{X} \mid X \in \mathfrak{P} \setminus S\}$. We will prove that T is a filter.

$1^{\mathfrak{P}} \in T$ because $0^{\mathfrak{P}} \notin S$; so T is nonempty. To prove that T is a filter it is enough to show $\forall X, Y \in \mathfrak{P}: (X, Y \in T \Leftrightarrow X \sqcap^{\mathfrak{P}} Y \in T)$. In fact,

$$\begin{aligned} X, Y \in T &\Leftrightarrow \\ \overline{X}, \overline{Y} \notin S &\Leftrightarrow \\ \neg(\overline{X} \in S \vee \overline{Y} \in S) &\Leftrightarrow \\ \overline{X} \sqcup^{\mathfrak{P}} \overline{Y} \notin S &\Leftrightarrow \\ \overline{\overline{X} \sqcup^{\mathfrak{P}} \overline{Y}} \in T &\Leftrightarrow \\ X \sqcap^{\mathfrak{P}} Y \in T. & \end{aligned}$$

So T is a filter. To finish the proof we will show that $\partial T = S$. In fact, for every $X \in \mathfrak{P}$

$$X \in \partial T \Leftrightarrow \overline{X} \notin \operatorname{up} T \Leftrightarrow \overline{X} \notin T \Leftrightarrow X \in S.$$

□

Proposition 4.130. If $\mathfrak{Z}$ is a boolean lattice then $\mathcal{A} \sqsubseteq \mathcal{B} \Leftrightarrow \partial\mathcal{A} \subseteq \partial\mathcal{B}$ for every $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.

Proof.

$$\begin{aligned} \partial\mathcal{A} \subseteq \partial\mathcal{B} &\Leftrightarrow \\ \{\overline{X} \mid X \in \mathfrak{P} \setminus \mathcal{A}\} \subseteq \{\overline{X} \mid X \in \mathfrak{P} \setminus \mathcal{B}\} &\Leftrightarrow \\ \mathfrak{P} \setminus \mathcal{A} \subseteq \mathfrak{P} \setminus \mathcal{B} &\Leftrightarrow \\ \mathcal{A} \supseteq \mathcal{B} &\Leftrightarrow \\ \mathcal{A} \sqsubseteq \mathcal{B}. & \end{aligned}$$

□

Corollary 4.131. ∂ is a straight monotone map if $\mathfrak{Z}$ is a boolean lattice.

Theorem 4.132. If $\mathfrak{Z}$ is a boolean lattice then $\partial \bigsqcup^{\mathfrak{F}} S = \bigcup \langle\partial\rangle S$ for every $S \in \mathscr{P}\mathfrak{F}$.

Proof. For boolean lattices ∂ is an order embedding from the poset $\mathfrak{F}$ to the complete lattice $\mathscr{P}\mathfrak{P}$. So accordingly the lemma 4.104 it is enough to prove that there exists $\mathcal{F} \in \mathfrak{F}$ such that $\partial\mathcal{F} = \bigcup \langle\partial\rangle S$. To prove this it is enough to show that $0^{\mathfrak{P}} \notin \bigcup \langle\partial\rangle S$ and

$$\forall A, B \in S: \left(A \sqcup^{\mathfrak{P}} B \in \bigcup \langle\partial\rangle S \Leftrightarrow A \in \bigcup \langle\partial\rangle S \vee B \in \bigcup \langle\partial\rangle S\right).$$

$0^{\mathfrak{P}} \notin \bigcup \langle\partial\rangle S$ is obvious.