Proof. Taking into account the previous section, we have:

$$\begin{aligned}
\mathcal{A}\sqcup^{\mathfrak{F}}\bigsqcap^{\mathfrak{F}} S &= \\
\mathcal{A}\cap\bigsqcap^{\mathfrak{F}} S &= \\
\mathcal{A}\cap\left\{K_0\sqcap^{\mathfrak{Z}}\ldots\sqcap^{\mathfrak{Z}}K_n \mid K_i\in\bigcup S \text{ where } i=0,\ldots,n \text{ for } n\in\mathbb{N}\right\} &= \\
\left\{K_0\sqcap^{\mathfrak{Z}}\ldots\sqcap^{\mathfrak{Z}}K_n \mid K_0\sqcap^{\mathfrak{Z}}\ldots\sqcap^{\mathfrak{Z}}K_n\in\mathcal{A}, K_i\in\bigcup S \text{ where } i=0,\ldots,n \text{ for } n\in\mathbb{N}\right\} &= \\
\left\{K_0\sqcap^{\mathfrak{Z}}\ldots\sqcap^{\mathfrak{Z}}K_n \mid K_i\in\mathcal{A}\wedge K_i\in\bigcup S \text{ where } i=0,\ldots,n \text{ for } n\in\mathbb{N}\right\} &= \\
\left\{K_0\sqcap^{\mathfrak{Z}}\ldots\sqcap^{\mathfrak{Z}}K_n \mid K_i\in\mathcal{A}\cap\bigcup S \text{ where } i=0,\ldots,n \text{ for } n\in\mathbb{N}\right\} &= \\
\left\{K_0\sqcap^{\mathfrak{Z}}\ldots\sqcap^{\mathfrak{Z}}K_n \mid K_i\in\bigcup\langle\mathcal{A}\cap\rangle S \text{ where } i=0,\ldots,n \text{ for } n\in\mathbb{N}\right\} &= \\
\left\{K_0\sqcap^{\mathfrak{Z}}\ldots\sqcap^{\mathfrak{Z}}K_n \mid K_i\in\bigcup\{\mathcal{A}\cap\mathcal{X}\mid\mathcal{X}\in S\} \text{ where } i=0,\ldots,n \text{ for } n\in\mathbb{N}\right\} &= \\
\left\{K_0\sqcap^{\mathfrak{Z}}\ldots\sqcap^{\mathfrak{Z}}K_n \mid K_i\in\bigcup\{\mathcal{A}\sqcup^{\mathfrak{F}}\mathcal{X}\mid\mathcal{X}\in S\} \text{ where } i=0,\ldots,n \text{ for } n\in\mathbb{N}\right\} &= \\
\bigsqcap^{\mathfrak{F}}\{\mathcal{A}\sqcup^{\mathfrak{F}}\mathcal{X}\mid\mathcal{X}\in S\} &= \\
\bigsqcap^{\mathfrak{F}}\langle\mathcal{A}\sqcup^{\mathfrak{F}}\rangle S.
\end{aligned}$$

□

Corollary 4.114. If $\mathfrak{Z}$ is a distributive lattice with greatest element, then $\mathfrak{F}$ is also a distributive lattice.

Corollary 4.115. If $\mathfrak{Z}$ is a distributive lattice with greatest element, then $\mathfrak{F}$ is a co-brouwerian lattice.

4.3.12 Filters over Boolean Lattices

Theorem 4.116. If $\mathfrak{Z}$ is a boolean lattice then $a\setminus^{\mathfrak{F}}B=a\sqcap^{\mathfrak{F}}\overline{B}$ for every $a\in\mathfrak{F}$, $B\in\mathfrak{P}$ (where the complement is taken on $\mathfrak{P}$).

Proof. $\mathfrak{F}$ is a distributive lattice by corollary 4.114. Our filtrator is finitely meet-closed by the theorem 4.44 and with join-closed core by the theorem 4.25. It is also up and down aligned.

So we can apply the proposition 4.74. □

4.3.12.1 Distributivity for an Element of Boolean Core

Lemma 4.117. Let $\mathfrak{F}$ be the poset of filters over a boolean lattice $\mathfrak{Z}$.

Then $A\sqcap^{\mathfrak{F}}$ is a lower adjoint of $\overline{A}\sqcup^{\mathfrak{F}}$ for every $A\in\mathfrak{P}$.

Proof. Lemma 4.75. □

Theorem 4.118. Let $\mathfrak{F}$ be the poset of filters over a boolean lattice $\mathfrak{Z}$. Then $A\sqcap^{\mathfrak{F}}\bigsqcup^{\mathfrak{F}} S=\bigsqcup^{\mathfrak{F}}\langle A\sqcap^{\mathfrak{F}}\rangle S$ for every $A\in\mathfrak{P}$ and every set $S\in\mathscr{P}\mathfrak{F}$.

Proof. Direct consequence of the lemma. □

4.3.13 Generalized Filter Base

Definition 4.119. *Generalized filter base* is a filter base on the set $\mathfrak{F}$.

Definition 4.120. If S is a generalized filter base and $\mathcal{A}=\bigsqcap^{\mathfrak{F}} S$, then we call S a generalized filter base of a filter $\mathcal{A}$.

Theorem 4.121. If $\mathfrak{Z}$ is a distributive lattice [TODO: Can be generalized for any meet-semilattice?] and S is a generalized filter base of a filter $\mathcal{F}$ then for any $K\in\mathfrak{Z}$

$$K\in\mathcal{F}\Leftrightarrow\exists\mathcal{L}\in S: K\in\mathcal{L}.$$