$X_i \in P_i$ for some $P_i \in S$; $C \sqcup^{\mathfrak{Z}} X_i \in P_i$; $C \sqcup^{\mathfrak{Z}} X_i \in \bigcup S$; consequently $C \in R$.

We have proved that that R is a filter base and an upper set. So R is a filter.

Let $\mathcal{A} \in S$. Then $\mathcal{A} \subseteq \bigcup S$;

$$R \supseteq \{K_0 \sqcap^{\mathfrak{Z}} \ldots \sqcap^{\mathfrak{Z}} K_n \mid K_i \in \mathcal{A} \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}\} = \mathcal{A}.$$

Consequently $\mathcal{A} \sqsupseteq R$.

Let now $\mathcal{B} \in \mathfrak{F}$ and $\forall \mathcal{A} \in S: \mathcal{A} \sqsupseteq \mathcal{B}$. Then $\forall \mathcal{A} \in S: \mathcal{A} \subseteq \mathcal{B}$; $\mathcal{B} \supseteq \bigcup S$. From this $\mathcal{B} \supseteq T$ for every finite set $T \subseteq \bigcup S$. Consequently $\mathcal{B} \ni \bigsqcap^{\mathfrak{Z}} T$. Thus $\mathcal{B} \supseteq R$; $\mathcal{B} \sqsubseteq R$.

Comparing we get $\bigsqcap^{\mathfrak{F}} S = R$. □

Theorem 4.111. If $\mathfrak{Z}$ is a distributive lattice then for any $\mathcal{F}_0, \ldots, \mathcal{F}_m \in \mathfrak{F}$ $(m \in \mathbb{N})$

$$\mathcal{F}_0 \sqcap^{\mathfrak{F}} \ldots \sqcap^{\mathfrak{F}} \mathcal{F}_m = \{K_0 \sqcap^{\mathfrak{Z}} \ldots \sqcap^{\mathfrak{Z}} K_m \mid K_i \in \mathcal{F}_i \text{ where } i = 0, \ldots, m\}.$$

Proof. Let's denote the right part of the equality to be proven as R. First we will prove that R is a filter. Obviously R is nonempty.

Let $A, B \in R$. Then $A = X_0 \sqcap^{\mathfrak{Z}} \ldots \sqcap^{\mathfrak{Z}} X_m$, $B = Y_0 \sqcap^{\mathfrak{Z}} \ldots \sqcap^{\mathfrak{Z}} Y_m$ where $X_i, Y_i \in \mathcal{F}_i$.

$$A \sqcap^{\mathfrak{Z}} B = (X_0 \sqcap^{\mathfrak{Z}} Y_0) \sqcap^{\mathfrak{Z}} \ldots \sqcap^{\mathfrak{Z}} (X_m \sqcap^{\mathfrak{Z}} Y_m),$$

consequently $A \sqcap^{\mathfrak{Z}} B \in R$.

Let filter $C \sqsupseteq A \in R$

$$C = A \sqcup^{\mathfrak{Z}} C = (X_0 \sqcup^{\mathfrak{Z}} C) \sqcap^{\mathfrak{Z}} \ldots \sqcap^{\mathfrak{Z}} (X_m \sqcup^{\mathfrak{Z}} C) \in R.$$

So R is a filter.

Let $P_i \in \mathcal{F}_i$. Then $P_i \in R$ because $P_i = (P_i \sqcup^{\mathfrak{Z}} P_0) \sqcap^{\mathfrak{Z}} \ldots \sqcap^{\mathfrak{Z}} (P_i \sqcup^{\mathfrak{Z}} P_m)$. So $\mathcal{F}_i \subseteq R$; $\mathcal{F}_i \sqsupseteq R$.

Let now $\mathcal{B} \in \mathfrak{F}$ and $\forall i \in \{0, \ldots, m\}: \mathcal{F}_i \sqsupseteq \mathcal{B}$. Then $\forall i \in \{0, \ldots, m\}: \mathcal{F}_i \subseteq \mathcal{B}$.

$L_i \in \mathcal{B}$ for every $L_i \in \mathcal{F}_i$. $L_0 \sqcap^{\mathfrak{Z}} \ldots \sqcap^{\mathfrak{Z}} L_m \in \mathcal{B}$. So $\mathcal{B} \supseteq R$; $\mathcal{B} \sqsubseteq R$.

So $\mathcal{F}_0 \sqcap^{\mathfrak{F}} \ldots \sqcap^{\mathfrak{F}} \mathcal{F}_m = R$. □

4.3.10 Separability of Core for Primary Filtrators

Theorem 4.112. A primary filtrator with least element, whose core is a distributive lattice, is with separable core. [TODO: Is distributivity necessary? I suspect it for every meet-semilattice.]

Proof. Let $\mathcal{A} \asymp^{\mathfrak{F}} \mathcal{B}$ where $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.

$$\mathcal{A} \sqcap^{\mathfrak{F}} \mathcal{B} = \{A \sqcap^{\mathfrak{Z}} B \mid A \in \mathcal{A}, B \in \mathcal{B}\}.$$

So

$$\begin{aligned} 0 \in \mathcal{A} \sqcap^{\mathfrak{F}} \mathcal{B} &\Leftrightarrow \\ \exists A \in \mathcal{A}, B \in \mathcal{B}: A \sqcap^{\mathfrak{Z}} B = 0 &\Leftrightarrow \\ \exists A \in \mathcal{A}, B \in \mathcal{B}: \uparrow A \sqcap^{\mathfrak{P}} \uparrow B = 0^{\mathfrak{F}} &\Leftrightarrow \\ \exists A \in \mathcal{A}, B \in \mathcal{B}: \uparrow A \sqcap^{\mathfrak{F}} \uparrow B = 0^{\mathfrak{F}} &\Leftrightarrow \\ \exists A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}: A \sqcap^{\mathfrak{F}} B = 0^{\mathfrak{F}} & \end{aligned}$$

(used proposition 4.97). □

4.3.11 Distributivity of the Lattice of Filters

Theorem 4.113. If $\mathfrak{Z}$ is a distributive lattice with greatest element, $S \in \mathscr{P}\mathfrak{F}$ and $\mathcal{A} \in \mathfrak{F}$ then [TODO: Can it be generalized for meet-semilattices (use generalized infinite meet formula in rewrite-plan.pdf)? Also corollaries.]

$$\mathcal{A} \sqcup^{\mathfrak{F}} \bigsqcap^{\mathfrak{F}} S = \bigsqcap^{\mathfrak{F}} \langle \mathcal{A} \sqcup^{\mathfrak{F}} \rangle S.$$