4.3.9 Formulas for Meets and Joins of Filters

Lemma 4.104. If f is an order embedding from a poset $\mathfrak{A}$ to a complete lattice $\mathfrak{B}$ and $S \in \mathscr{P}\mathfrak{A}$ and there exists such $\mathcal{F} \in \mathfrak{A}$ that $f\mathcal{F} = \bigsqcup^{\mathfrak{B}} \langle f \rangle S$, then $\bigsqcup^{\mathfrak{A}} S$ exists and $f \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{B}} \langle f \rangle S$.

Proof. f is an order isomorphism from $\mathfrak{A}$ to $\mathfrak{B}|_{\langle f \rangle \mathfrak{A}}$. $f\mathcal{F} \in \mathfrak{B}|_{\langle f \rangle \mathfrak{A}}$.

Consequently, $\bigsqcup^{\mathfrak{B}} \langle f \rangle S \in \mathfrak{B}|_{\langle f \rangle \mathfrak{A}}$ and $\bigsqcup^{\mathfrak{B}|_{\langle f \rangle \mathfrak{A}}} \langle f \rangle S = \bigsqcup^{\mathfrak{B}} \langle f \rangle S$.

$f \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{B}|_{\langle f \rangle \mathfrak{A}}} \langle f \rangle S$ because f is an order isomorphism.

Combining, $f \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{B}} \langle f \rangle S$. □

Corollary 4.105. If $\mathfrak{B}$ is a complete lattice and $\mathfrak{A}$ is its subset and $S \in \mathscr{P}\mathfrak{A}$ and $\bigsqcup^{\mathfrak{B}} S \in \mathfrak{A}$, then $\bigsqcup^{\mathfrak{A}} S$ exists and $\bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{B}} S$.

Theorem 4.106. If $\mathfrak{Z}$ is a meet-semilattice with greatest element 1 then $\bigsqcup^{\mathfrak{F}} S$ exists and

$$\bigsqcup^{\mathfrak{F}} S = \bigcap S$$

for every $S \in \mathscr{P}\mathfrak{F}$.

Proof. Taking into account the corollary of the lemma, it is enough to prove that there exists $\mathcal{F} \in \mathfrak{F}$ such that $\mathcal{F} = \bigcap S$, that is that $R = \bigcap S$ is a filter.

R is nonempty because $1 \in R$. Let $A, B \in R$; then $\forall \mathcal{F} \in S$: $A, B \in \mathcal{F}$, consequently $\forall \mathcal{F} \in S$: $A \sqcap^{\mathfrak{Z}} B \in \mathcal{F}$. Consequently $A \sqcap^{\mathfrak{Z}} B \in \bigcap S = R$. So R is a filter base. Let $X \in R$ and $X \sqsubseteq Y \in \mathfrak{Z}$; then $\forall \mathcal{F} \in S$: $X \in \mathcal{F}$; $\forall \mathcal{F} \in S$: $Y \in \mathcal{F}$; $Y \in R$. So R is an upper set. □

Corollary 4.107. If $\mathfrak{Z}$ is a meet-semilattice with greatest element then $\mathfrak{F}$ is a complete lattice.

Corollary 4.108. If $\mathfrak{Z}$ is a meet-semilattice with greatest element then for any $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$

$$\mathcal{A} \sqcup^{\mathfrak{F}} \mathcal{B} = \mathcal{A} \cap \mathcal{B}.$$

We will denote meets and joins on the lattice of filters just as $\sqcap$ and $\sqcup$.

Theorem 4.109. If $\mathfrak{Z}$ is a join-semilattice then $\mathfrak{F}$ is a join-semilattice and for any $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$

$$\mathcal{A} \sqcup^{\mathfrak{F}} \mathcal{B} = \mathcal{A} \cap \mathcal{B}.$$

Proof. Taking into account the corollary of the lemma, it is enough to prove $R = \mathcal{A} \cap \mathcal{B}$ is a filter.

R is nonempty because there exists $X \in \mathcal{A}$ and $Y \in \mathcal{B}$ and $R \ni X \sqcup^{\mathfrak{Z}} Y$.

Let $A, B \in R$. Then $A, B \in \mathcal{A}$; so there exists $C \in \mathcal{A}$ such that $C \sqsubseteq A \wedge C \sqsubseteq B$. Analogously there exists $D \in \mathcal{B}$ such that $D \sqsubseteq A \wedge D \sqsubseteq B$. Let $E = C \sqcup^{\mathfrak{Z}} D$. Then $E \in \mathcal{A}$ and $E \in \mathcal{B}$; $E \in R$ and $E \sqsubseteq A \wedge E \sqsubseteq B$. So R is a filter base.

That R is an upper set is obvious. □

Theorem 4.110. If $\mathfrak{Z}$ is a distributive lattice then for $S \in \mathscr{P}\mathfrak{F} \setminus \{\emptyset\}$

$$\bigsqcap^{\mathfrak{F}} S = \left\{ K_0 \sqcap^{\mathfrak{Z}} \ldots \sqcap^{\mathfrak{Z}} K_n \mid K_i \in \bigcup S \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N} \right\}.$$

Proof. Let's denote the right part of the equality to be proven as R. First we will prove that R is a filter. R is nonempty because S is nonempty.

Let $A, B \in R$. Then $A = X_0 \sqcap^{\mathfrak{Z}} \ldots \sqcap^{\mathfrak{Z}} X_k$, $B = Y_0 \sqcap^{\mathfrak{Z}} \ldots \sqcap^{\mathfrak{Z}} Y_l$ where $X_i, Y_j \in \bigcup S$. So

$$A \sqcap^{\mathfrak{Z}} B = X_0 \sqcap^{\mathfrak{Z}} \ldots \sqcap^{\mathfrak{Z}} X_k \sqcap^{\mathfrak{Z}} Y_0 \sqcap^{\mathfrak{Z}} \ldots \sqcap^{\mathfrak{Z}} Y_l \in R.$$

Let filter $C \sqsupseteq A \in R$. Consequently (distributivity used)

$$C = C \sqcup^{\mathfrak{Z}} A = (C \sqcup^{\mathfrak{Z}} X_0) \sqcap^{\mathfrak{Z}} \ldots \sqcap^{\mathfrak{Z}} (C \sqcup^{\mathfrak{Z}} X_k).$$