Let $X \sqcap Y \in F$; then $X, Y \in F$ because F is an upper set.

(3)⇒(2). Let

$$\forall X, Y \in \mathfrak{Z}: (X, Y \in F \Leftrightarrow X \sqcap Y \in F).$$

Then $\forall X, Y \in F: X \sqcap Y \in F$. Let $X \in F$ and $X \sqsubseteq Y \in \mathfrak{Z}$. Then $X \sqcap Y = X \in F$. Consequently $X, Y \in F$. So F is an upper set. □

Proposition 4.83. Let S be a filter base on a meet-semilattice. If $A_0, \ldots, A_n \in S$ ($n \in \mathbb{N}$), then

$$\exists C \in S: C \sqsubseteq A_0 \sqcap \ldots \sqcap A_n.$$

Proof. It can be easily proved by induction. □

Proposition 4.84. If $\mathfrak{Z}$ is a meet-semilattice and S is a filter base on it, $A \in \mathfrak{Z}$, then $\langle A \sqcap \rangle S$ is also a filter base.

Proof. $\langle A \sqcap \rangle S \neq \emptyset$ because $S \neq \emptyset$.

Let $X, Y \in \langle A \sqcap \rangle S$. Then $X = A \sqcap X'$ and $Y = A \sqcap Y'$ where $X', Y' \in S$. There exists $Z' \in S$ such that $Z' \sqsubseteq X' \sqcap Y'$. So $X \sqcap Y = A \sqcap X' \sqcap Y' \sqsupseteq A \sqcap Z' \in \langle A \sqcap \rangle S$. □

4.3.3 Order of filters. Principal filters

I will make the set of filters $\mathfrak{F}$ into a poset by the order defined by the formula: $a \sqsubseteq b \Leftrightarrow a \supseteq b$.

Definition 4.85. The *principal filter* corresponding to an element $a \in \mathfrak{Z}$ is

$$\uparrow a = \{x \in \mathfrak{Z} \mid x \sqsupseteq a\}.$$

Elements of $\mathfrak{P} = \langle \uparrow \rangle \mathfrak{Z}$ are called *principal filters*.

Obvious 4.86. Principal filters are filters.

Obvious 4.87. $\uparrow$ is an order embedding from $\mathfrak{Z}$ to $\mathfrak{F}$.

Corollary 4.88. $\uparrow$ is an order isomorphism between $\mathfrak{Z}$ and $\mathfrak{P}$.

Definition 4.89. For every poset $\mathfrak{Z}$ I call $(\mathfrak{F}; \mathfrak{P})$ the *primary filtrator* (for the base $\mathfrak{Z}$).

Proposition 4.90. $\uparrow K \sqsupseteq \mathcal{A} \Leftrightarrow K \in \mathcal{A}$.

Proof. $\uparrow K \sqsupseteq \mathcal{A} \Leftrightarrow \uparrow K \subseteq \mathcal{A} \Leftrightarrow K \in \mathcal{A}$. □

Proposition 4.91. $\operatorname{up} a = \langle \uparrow \rangle a$ for an element a of a primary filtrator.

Proof. For every $L \in \mathfrak{P}$ we have $L = \uparrow K$ for some $K \in \mathfrak{Z}$ and $L \in \operatorname{up} a \Leftrightarrow L \sqsupseteq a \Leftrightarrow \uparrow K \sqsupseteq a \Leftrightarrow K \in a \Leftrightarrow L \in \langle \uparrow \rangle a$. □

4.3.3.1 Minimal and maximal filters

Obvious 4.92. The filter $0^{\mathfrak{F}} = \mathfrak{Z}$ (equal to the principal filter for the least element of $\mathfrak{Z}$ if it exists) is the least element of the poset of filters.

Proposition 4.93. If there exists greatest element $1^{\mathfrak{Z}}$ of the poset $\mathfrak{Z}$ then $1^{\mathfrak{F}} = \{1^{\mathfrak{Z}}\}$ is the greatest element of $\mathfrak{F}$.

Proof. Take into account that filters are nonempty. □

4.3.4 Primary filtrator is filtered

[TODO: Can the proof be simplified using the fact that “filtered” is the same as “semifiltered”?]