- **⇐.** We will assume that cardinality of a set is an ordinal defined by von Neumann cardinal assignment (what is a standard practice in ZFC). Recall that $\alpha<\beta\Leftrightarrow\alpha\in\beta$ for ordinals α, β.

  We will take it as given that for every nonempty chain $T\in\mathscr{P}S$ we have $\bigsqcap T\in S$.

  We will prove the following statement: If $\operatorname{card}S=n$ then S is filter closed, for any cardinal n.

  Instead we will prove it not only for cardinals but for wider class of ordinals: If $\operatorname{card}S=n$ then S is filter closed, for any ordinal n.

  We will prove it using transfinite induction by n.

  For finite n we have $\bigsqcap T\in S$ because $T\subseteq S$ has minimal element.

  Let $\operatorname{card}T=n$ be an infinite ordinal.

  Let the assumption hold for every $m\in\operatorname{card}T$.

  We can assign $T=\{a_\alpha\mid\alpha\in\operatorname{card}T\}$ for some a_α because $\operatorname{card}\operatorname{card}T=\operatorname{card}T$.

  Consider $\beta\in\operatorname{card}T$.

  Let $P_\beta=\{a_\alpha\mid\alpha\in\beta\}$. Let $b_\beta=\bigsqcap P_\beta$. Obviously $b_\beta=\bigsqcap[P_\beta]_\sqcap$. We have

  $$\operatorname{card}[P_\beta]_\sqcap=\operatorname{card}P_\beta=\operatorname{card}\beta<\operatorname{card}T$$

  (used the lemma and von Neumann cardinal assignment). By the assumption of induction $b_\beta\in S$.

  $\forall\beta\in\operatorname{card}T\colon P_\beta\subseteq T$ and thus $b_\beta\sqsupseteq\bigsqcap T$.

  It is easy to see that the set $\{P_\beta\mid\beta\in\operatorname{card}T\}$ is a chain. Consequently $\{b_\beta\mid\beta\in\operatorname{card}T\}$ is a chain.

  By the theorem conditions $b=\bigsqcap\{b_\beta\mid\beta\in\operatorname{card}T\}\in S$ (taken into account that $b_\beta\in S$ by the assumption of induction).

  Obviously $b\sqsupseteq\bigsqcap T$.

  $b\sqsubseteq b_\beta$ and so $\forall\beta\in\operatorname{card}T,\alpha\in\beta\colon b\sqsubseteq a_\alpha$. Let $\alpha\in\operatorname{card}T$. Then (because $\operatorname{card}T$ is limit ordinal, see [41]) there exists $\beta\in\operatorname{card}T$ such that $\alpha\in\beta\in\operatorname{card}T$. So $b\sqsubseteq a_\alpha$ for every $\alpha\in\operatorname{card}T$. Thus $b\sqsubseteq\bigsqcap T$.

  Finally $\bigsqcap T=b\in S$. □

4.2.9 Complements and Core Parts

Lemma 4.59. If $(\mathfrak{A};\mathfrak{Z})$ is a filtered, up-aligned filtrator with co-separable core which is a complete lattice, then for any $a,c\in\mathfrak{A}$

$$c\equiv^{\mathfrak{A}}a\Leftrightarrow c\equiv^{\mathfrak{A}}\operatorname{Cor}a.$$

Proof.

- **⇒.** If $c\equiv^{\mathfrak{A}}a$ then by co-separability of the core exists $K\in\operatorname{down}a$ such that $c\equiv^{\mathfrak{A}}K$. To finish the proof we will show that $K\sqsubseteq\operatorname{Cor}a$. To show this is enough to show that $\forall X\in\operatorname{up}a\colon K\sqsubseteq X$ what is obvious.
- **⇐.** $\operatorname{Cor}a\sqsubseteq a$ (by the theorem 4.29 using that our filtrator is filtered). □

Theorem 4.60. If $(\mathfrak{A};\mathfrak{Z})$ is a filtered up-aligned complete lattice filtrator with co-separable core which is a complete boolean lattice, then $a^+=\overline{\operatorname{Cor}a}$ for every $a\in\mathfrak{A}$.

Proof. Our filtrator is with join-closed core (theorem 4.25).

$$\begin{aligned}a^+ &=\\ \bigsqcap^{\mathfrak{A}}\{c\in\mathfrak{A}\mid c\sqcup^{\mathfrak{A}}a=1^{\mathfrak{A}}\} &=\\ \bigsqcap^{\mathfrak{A}}\{c\in\mathfrak{A}\mid c\sqcup^{\mathfrak{A}}\operatorname{Cor}a=1^{\mathfrak{A}}\} &=\\ \bigsqcap^{\mathfrak{A}}\{c\in\mathfrak{A}\mid c\sqsupseteq\overline{\operatorname{Cor}a}\} &=\\ \overline{\operatorname{Cor}a}&\end{aligned}$$