4.2.8 Some Criteria

Theorem 4.53. For a semifiltered, star-separable, down-aligned filtrator $(\mathfrak{A};\mathfrak{Z})$ with finitely meet closed and separable core where $\mathfrak{Z}$ is a complete boolean lattice and both $\mathfrak{Z}$ and $\mathfrak{A}$ are atomistic lattices the following conditions are equivalent for any $\mathcal{F}\in\mathfrak{A}$:

1. $\mathcal{F}\in\mathfrak{Z}$;
2. $\forall S\in\mathscr{P}\mathfrak{A}\colon \left(\mathcal{F}\sqcap^{\mathfrak{A}}\bigsqcup^{\mathfrak{A}} S\neq 0\Rightarrow\exists\mathcal{K}\in S\colon\mathcal{F}\sqcap^{\mathfrak{A}}\mathcal{K}\neq 0\right)$;
3. $\forall S\in\mathscr{P}\mathfrak{Z}\colon \left(\mathcal{F}\sqcap^{\mathfrak{A}}\bigsqcup^{\mathfrak{A}} S\neq 0\Rightarrow\exists K\in S\colon\mathcal{F}\sqcap^{\mathfrak{A}}K\neq 0\right)$.

Proof. Our filtrator is with join-closed core (theorem 4.25).

(1)⇒(2). Let $\mathcal{F}\in\mathfrak{Z}$. Then (taking into account the proposition 4.43)

$$\mathcal{F}\sqcap^{\mathfrak{A}}\bigsqcup^{\mathfrak{A}} S\neq 0\Leftrightarrow\overline{\mathcal{F}}\nsupseteq\bigsqcup^{\mathfrak{A}} S\Rightarrow\exists\mathcal{K}\in S\colon\overline{\mathcal{F}}\nsupseteq\mathcal{K}\Leftrightarrow\exists\mathcal{K}\in S\colon\mathcal{F}\sqcap^{\mathfrak{A}}\mathcal{K}\neq 0.$$

(2)⇒(3). Obvious.

(3)⇒(1). Let the formula (3) be true. Then for $L\in\mathfrak{Z}$ and $S=\operatorname{atoms}^{\mathfrak{Z}}L$ it takes the form

$$\mathcal{F}\sqcap^{\mathfrak{A}}\bigsqcup^{\mathfrak{A}}\operatorname{atoms}^{\mathfrak{Z}}L\neq 0\Rightarrow\exists K\in S\colon\mathcal{F}\sqcap^{\mathfrak{A}}K\neq 0$$

that is $\mathcal{F}\sqcap^{\mathfrak{A}}L\neq 0\Rightarrow\exists K\in S\colon\mathcal{F}\sqcap^{\mathfrak{A}}K\neq 0$ because $\bigsqcup^{\mathfrak{A}}\operatorname{atoms}^{\mathfrak{Z}}L=\bigsqcup^{\mathfrak{Z}}\operatorname{atoms}^{\mathfrak{Z}}L=L$. That is $\mathcal{F}\sqcap^{\mathfrak{A}}L\neq 0\Rightarrow\mathcal{F}\sqcap^{\mathfrak{A}}K_L\neq 0$ where $K_L\in S$. Thus K_L is an atom of both $\mathfrak{A}$ and $\mathfrak{Z}$ (see the theorem 4.49), so having $\mathcal{F}\sqcap^{\mathfrak{A}}L\neq 0\Rightarrow\mathcal{F}\sqsupseteq K_L$. Let

$$F=\bigsqcup^{\mathfrak{Z}}\{K_L\mid L\in\mathfrak{Z},\mathcal{F}\sqcap^{\mathfrak{A}}L\neq 0\}.$$

Then

$$F=\bigsqcup^{\mathfrak{A}}\{K_L\mid L\in\mathfrak{Z},\mathcal{F}\sqcap^{\mathfrak{A}}L\neq 0\}.$$

Obviously $F\sqsubseteq\mathcal{F}$. We have $L\sqcap^{\mathfrak{A}}\mathcal{F}\neq 0\Rightarrow K_L\sqcap^{\mathfrak{Z}}F\neq 0\Rightarrow L\sqcap^{\mathfrak{Z}}F\neq 0\Rightarrow L\sqcap^{\mathfrak{A}}F\neq 0$, thus by star separability of our filtrator $\mathcal{F}\sqsubseteq F$ and so $\mathcal{F}=F\in\mathfrak{Z}$. □

Definition 4.54. Let S be a subset of a meet-semilattice. The *filter base generated by* S is the set

$$[S]_{\sqcap}=\{a_0\sqcap\ldots\sqcap a_n\mid a_i\in S,i=0,1,\ldots\}.$$

Lemma 4.55. The set of all finite subsets of an infinite set A has the same cardinality as A.

Proof. Let denote the number of n-element subsets of A as s_n. Obviously $s_n\leqslant\operatorname{card}A^n=\operatorname{card}A$. Then the number S of all finite subsets of A is equal to $s_0+s_1+\ldots\leqslant\operatorname{card}A+\operatorname{card}A+\ldots=\operatorname{card}A$. That $S\geqslant\operatorname{card}A$ is obvious. So $S=\operatorname{card}A$. □

Lemma 4.56. A filter base generated by an infinite set has the same cardinality as that set.

Proof. From the previous lemma. □

Definition 4.57. Let $\mathfrak{A}$ be a complete lattice. A set $S\in\mathscr{P}\mathfrak{A}$ is *filter-closed* when for every filter base $T\in\mathscr{P}S$ we have $\bigsqcap T\in S$.

Theorem 4.58. A subset S of a complete lattice is filter-closed iff for every nonempty chain $T\in\mathscr{P}S$ we have $\bigsqcap T\in S$.

Proof. (proof sketch by Joel David Hamkins)

⇒. Because every nonempty chain is a filter base.