Definition 4.48. I call a filtrator *star-separable* when its core is a separation subset of its base.

4.2.6 Atomic Elements of a Filtrator

See [4] and [9] for more detailed treatment of ultrafilters and prime filters.

Theorem 4.49. Let $(\mathfrak{A};\mathfrak{Z})$ be a semifiltered down-aligned filtrator with finitely meet-closed core $\mathfrak{Z}$ which is a meet-semilattice. Then a is an atom of $\mathfrak{Z}$ iff $a\in\mathfrak{Z}$ and a is an atom of $\mathfrak{A}$.

Proof.

⇐. Obvious.

⇒. We need to prove that if a is an atom of $\mathfrak{Z}$ then a is an atom of $\mathfrak{A}$. Suppose the contrary that a is not an atom of $\mathfrak{A}$. Then there exists $x\in\mathfrak{A}$ such that $0\neq x\sqsubset a$. Because "up" is a straight monotone map to the dual of the poset $\mathscr{P}\mathfrak{Z}$ (obvious 4.24), $\operatorname{up} a\subset\operatorname{up} x$. So there exists $K\in\operatorname{up} x$ such that $K\notin\operatorname{up} a$. Also $a\in\operatorname{up} x$. We have $K\sqcap^{\mathfrak{Z}} a=K\sqcap^{\mathfrak{A}} a\in\operatorname{up} x$; $K\sqcap^{\mathfrak{Z}} a\neq 0$ and $K\sqcap^{\mathfrak{Z}} a\sqsubset a$. So a is not an atom of $\mathfrak{Z}$. □

Theorem 4.50. Let $(\mathfrak{A};\mathfrak{Z})$ be a semifiltered down-aligned filtrator and $\mathfrak{A}$ is a meet-semilattice. Then $a\in\mathfrak{A}$ is an atom of $\mathfrak{A}$ iff $\operatorname{up} a=\partial a$.

Proof.

⇒. Let a be an atom of $\mathfrak{A}$. $\operatorname{up} a\supseteq\partial a$ because $a\neq 0$. $\operatorname{up} a\subseteq\partial a$ because for any $K\in\mathfrak{A}$

$$K\in\operatorname{up} a\Leftrightarrow K\sqsupseteq a\Leftrightarrow K\sqcap^{\mathfrak{A}} a\neq 0\Leftrightarrow K\in\partial a.$$

⇐. Let $\operatorname{up} a=\partial a$. Then $a\neq 0$. Consequently for every $x\in\mathfrak{A}$ we have

$$\begin{aligned} 0\sqsubset x\sqsubset a &\Rightarrow\\ x\sqcap^{\mathfrak{A}} a\neq 0 &\Rightarrow\\ \forall K\in\operatorname{up} x: K\in\partial a &\Rightarrow\\ \forall K\in\operatorname{up} x: K\in\operatorname{up} a &\Rightarrow\\ \operatorname{up} x\subseteq\operatorname{up} a &\Rightarrow\\ x\sqsupseteq a. & \end{aligned}$$

So a is an atom of $\mathfrak{A}$. □

4.2.7 Prime Filtrator Elements

Definition 4.51. Let $(\mathfrak{A};\mathfrak{Z})$ be a down-aligned filtrator. *Prime* filtrator elements are such $a\in\mathfrak{A}$ that $\operatorname{up} a$ is a free star.

Proposition 4.52. Let $(\mathfrak{A};\mathfrak{Z})$ be a down-aligned filtrator with finitely join-closed core, where $\mathfrak{A}$ is a starrish join-semilattice and $\mathfrak{Z}$ is a join-semilattice. Then atomic elements of this filtrator are prime.

Proof. Let a be an atom of the lattice $\mathfrak{A}$. We have for every $X,Y\in\mathfrak{Z}$

$$\begin{aligned} X\sqcup^{\mathfrak{Z}} Y\in\operatorname{up} a &\Leftrightarrow\\ X\sqcup^{\mathfrak{A}} Y\in\operatorname{up} a &\Leftrightarrow\\ X\sqcup^{\mathfrak{A}} Y\sqsupseteq a &\Leftrightarrow\\ X\sqcup^{\mathfrak{A}} Y\not\asymp^{\mathfrak{A}} a &\Leftrightarrow\\ X\not\asymp^{\mathfrak{A}} a\vee Y\not\asymp^{\mathfrak{A}} a &\Leftrightarrow\\ X\sqsupseteq a\vee Y\sqsupseteq a &\Leftrightarrow\\ X\in\operatorname{up} a\vee Y\in\operatorname{up} a. & \end{aligned}$$

□