2. $B \equiv^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \overline{B} \sqsubseteq \mathcal{A}$ if it is up-aligned, with finitely join-closed and co-separable core.

Proof. We will prove only the first as the second is dual.

$$\begin{aligned} B \asymp^{\mathfrak{A}} \mathcal{A} &\Leftrightarrow \\ \exists A \in \operatorname{up} \mathcal{A}: B \asymp^{\mathfrak{A}} A &\Leftrightarrow \\ \exists A \in \operatorname{up} \mathcal{A}: B \sqcap^{\mathfrak{A}} A = 0 &\Leftrightarrow \\ \exists A \in \operatorname{up} \mathcal{A}: B \sqcap^{\mathfrak{Z}} A = 0 &\Leftrightarrow \\ \exists A \in \operatorname{up} \mathcal{A}: \overline{B} \sqsupseteq A &\Leftrightarrow \\ \overline{B} \in \operatorname{up} \mathcal{A} &\Leftrightarrow \\ \overline{B} \sqsupseteq \mathcal{A}. & \end{aligned}$$

□

4.2.4 Characterization of Finitely Meet-Closed Filtrators

Theorem 4.44. The following are equivalent for a filtrator $(\mathfrak{A}; \mathfrak{Z})$ whose core is a meet semilattice such that $\forall a \in \mathfrak{A}: \operatorname{up} a \neq \emptyset$:

1. The filtrator is finitely meet-closed.
2. $\operatorname{up} a$ is a filter for every $a \in \mathfrak{A}$.

Proof.

(1)⇒(2). Let $X, Y \in \operatorname{up} a$. Then $X \sqcap^{\mathfrak{Z}} Y = X \sqcap^{\mathfrak{A}} Y \sqsupseteq a$. That $\operatorname{up} a$ is an upper set is obvious. So taking into account that $\operatorname{up} a \neq \emptyset$, $\operatorname{up} a$ is a filter.

(2)⇒(1). It is enough to prove that $a \sqsubseteq A, B \Rightarrow a \sqsubseteq A \sqcap^{\mathfrak{Z}} B$ for every $A, B \in \mathfrak{A}$. Really:

$$a \sqsubseteq A, B \Rightarrow A, B \in \operatorname{up} a \Rightarrow A \sqcap^{\mathfrak{Z}} B \in \operatorname{up} a \Rightarrow a \sqsubseteq A \sqcap^{\mathfrak{Z}} B.$$ □

4.2.5 Stars of Elements of Filtrators

Definition 4.45. Let $(\mathfrak{A}; \mathfrak{Z})$ be a filtrator. *Core star* of an element a of a filtrator is

$$\partial a = \{x \in \mathfrak{Z} \mid x \not\asymp^{\mathfrak{A}} a\}.$$

Proposition 4.46. $\operatorname{up} a \subseteq \partial a$ for any non-least element a of a filtrator.

Proof. For any element $X \in \mathfrak{Z}$

$$X \in \operatorname{up} a \Rightarrow a \sqsubseteq X \wedge a \sqsubseteq a \Rightarrow X \not\asymp^{\mathfrak{A}} a \Rightarrow X \in \partial a.$$ □

Theorem 4.47. Let $(\mathfrak{A}; \mathfrak{Z})$ be a distributive lattice filtrator with least element and finitely join-closed core which is a join semilattice. Then ∂a is a free star for each $a \in \mathfrak{A}$.

Proof. For every $A, B \in \mathfrak{Z}$

$$\begin{aligned} A \sqcup^{\mathfrak{Z}} B \in \partial a &\Leftrightarrow \\ A \sqcup^{\mathfrak{A}} B \in \partial a &\Leftrightarrow \\ (A \sqcup^{\mathfrak{A}} B) \sqcap^{\mathfrak{A}} a \neq 0^{\mathfrak{A}} &\Leftrightarrow \\ (A \sqcap^{\mathfrak{A}} a) \sqcup^{\mathfrak{A}} (B \sqcap^{\mathfrak{A}} a) \neq 0^{\mathfrak{A}} &\Leftrightarrow \\ A \sqcap^{\mathfrak{A}} a \neq 0^{\mathfrak{A}} \vee B \sqcap^{\mathfrak{A}} a \neq 0^{\mathfrak{A}} &\Leftrightarrow \\ A \in \partial a \vee B \in \partial a. & \end{aligned}$$

That ∂a doesn't contain $0^{\mathfrak{A}}$ is obvious. □