Exercise 4.2. Describe "the neighborhood of positive infinity" filter on $\mathbb{R}$.

Definition 4.2. A filter not containing empty set is called a *proper filter*.

Obvious 4.3. The non-proper filter is $\mathscr{P}\mho$.

Remark 4.4. Some other authors require that all filters are proper. This is a stupid idea and we allow non-proper filters, in the same way as we allow to use the number 0.

4.1.2 Intro to filters on a meet-semilattice

A trivial generalization of the above:

Definition 4.5. A filter on a meet-semilattice $\mathfrak{Z}$ is a $\mathcal{F} \in \mathscr{P}\mathfrak{Z}$ such that:

1. $\forall A, B \in \mathcal{F}: A \sqcap B \in \mathcal{F}$;
2. $\forall A, B \in \mathfrak{Z}: (A \in \mathcal{F} \wedge B \sqsupseteq A \Rightarrow B \in \mathcal{F})$.

4.1.3 Intro to filters on a poset

Definition 4.6. A filter on a poset $\mathfrak{Z}$ is a $\mathcal{F} \in \mathscr{P}\mathfrak{Z}$ such that:

1. $\forall A, B \in \mathcal{F} \exists C \in \mathcal{F}: C \sqsubseteq A, B$;
2. $\forall A, B \in \mathfrak{Z}: (A \in \mathcal{F} \wedge B \sqsupseteq A \Rightarrow B \in \mathcal{F})$.

It is easy to show (and there is a proof of it somewhere below) that this coincides with the above definition in the case if $\mathfrak{Z}$ is a meet-semilattice.

4.1.4 Intro to filtrators

Definition 4.7. Filter $\uparrow x = \{c \in \mathfrak{Z} \mid c \sqsupseteq x\}$ is called the *principal filter induced by* the element x. A filter is *principal* iff it is a principal filter induced by some element.

I denote $\mathfrak{P}$ the set of all principal filters (for a given poset $\mathfrak{Z}$).

Now let (only in this paragraph) $\mathfrak{F}$ is an arbitrary poset and $\mathfrak{P}$ is its subset. I call pairs $(\mathfrak{F}; \mathfrak{P})$ of a poset with its subset *filtrators*. And when $\mathfrak{F}$ is the set of filters and $\mathfrak{P}$ is the set of principal filters on some poset I call them *primary filtrators*.

Filtrators are a more general case than the special case of filtrators on powersets.

4.2 Filtrators

Definition 4.8. I will call a *filtrator* a pair $(\mathfrak{A}; \mathfrak{Z})$ of a poset $\mathfrak{A}$ and its subset $\mathfrak{Z} \subseteq \mathfrak{A}$. I call $\mathfrak{A}$ the *base* of the filtrator and $\mathfrak{Z}$ the *core* of the filtrator. I will also say that $(\mathfrak{A}; \mathfrak{Z})$ is a filtrator *over* poset $\mathfrak{Z}$.

Definition 4.9. I will call a *lattice filtrator* a pair $(\mathfrak{A}; \mathfrak{Z})$ of a lattice $\mathfrak{A}$ and its subset $\mathfrak{Z} \subseteq \mathfrak{A}$.

Definition 4.10. I will call a *complete lattice filtrator* a pair $(\mathfrak{A}; \mathfrak{Z})$ of a complete lattice $\mathfrak{A}$ and its subset $\mathfrak{Z} \subseteq \mathfrak{A}$.

Definition 4.11. I will call a *central filtrator* a filtrator $(\mathfrak{A}; Z(\mathfrak{A}))$ where $Z(\mathfrak{A})$ is the center of a bounded lattice $\mathfrak{A}$.

Definition 4.12. I will call *element* of a filtrator an element of its base.

Definition 4.13. $\operatorname{up} a = \{c \in \mathfrak{Z} \mid c \sqsupseteq a\}$ for an element a of a filtrator.