Proof. From the previous. □

Remark 3.100. Note that the above formulas contain both $\bigcup_{i\in\operatorname{dom}F}\operatorname{dom}F_i'$ and $\bigcup_{i\in\operatorname{dom}F}F_i'$. These forms are similar but different.

3.8.4.7 Associativity of ordinated product

Let f be an ordinal variadic function.

Let S be an ordinal indexed family of functions of ordinal indexed families of functions each taking an ordinal number of arguments in a set X.

I call f *infinite associative* when

1. $f(f\circ S)=f(\operatorname{concat}S)$ for every S;
2. $f(\llbracket x\rrbracket)=x$ for $x\in X$.

Infinite associativity implies associativity

Proposition 3.101. Let f be an infinitely associative function taking an ordinal number of arguments in a set X. Define $x\star y=f\llbracket x;y\rrbracket$ for $x,y\in X$. Then the binary operation $\star$ is associative.

Proof. Let $x,y,z\in X$. Then $(x\star y)\star z=f\llbracket f\llbracket x;y\rrbracket;z\rrbracket=f(f\llbracket x;y\rrbracket;f\llbracket z\rrbracket)=f\llbracket x;y;z\rrbracket$. Similarly $x\star(y\star z)=f\llbracket x;y;z\rrbracket$. So $(x\star y)\star z=x\star(y\star z)$. □

Concatenation is associative

First we will prove some lemmas.

Let a and b be functions on a poset. Let $a\sim b$ iff there exist an order isomorphism f such that $a=b\circ f$. Evidently $\sim$ is an equivalence relation.

Obvious 3.102. $\operatorname{concat}a=\operatorname{concat}b\Leftrightarrow\operatorname{uncurry}(a)\sim\operatorname{uncurry}(b)$ for every ordinal indexed families a and b of functions taking an ordinal number of arguments.

Thank to the above, we can reduce properties of concat to properties of uncurry.

Lemma 3.103. $a\sim b\Rightarrow\operatorname{uncurry}a\sim\operatorname{uncurry}b$ for every ordinal indexed families a and b of functions taking an ordinal number of arguments.

Proof. There exist an order isomorphism f such that $a=b\circ f$.

$\operatorname{uncurry}(a)(x;y)=(ax)y=(bfx)y=\operatorname{uncurry}(b)(fx;y)=\operatorname{uncurry}(b)g(x;y)$ where $g(x;y)=(fx;y)$.

g is an order isomorphism because $g(x_0;y_0)\geqslant g(x_1;y_1)\Leftrightarrow(x_0;y_0)\geqslant(x_1;y_1)$. (Injectivity and surjectivity are obvious.) □

Lemma 3.104. Let $a_i\sim b_i$ for some f_i for every i. Then $\operatorname{uncurry}a\sim\operatorname{uncurry}b$ for every ordinal indexed families a and b of ordinal indexed families of functions taking an ordinal number of arguments.

Proof. Let $a_i=b_i\circ f_i$ where f_i is an order isomorphism for every i.

$\operatorname{uncurry}(a)(i;y)=a_i\,y=b_i\,f_i\,y=\operatorname{uncurry}(b)(i;f_i\,y)=\operatorname{uncurry}(b)g(i;y)=(\operatorname{uncurry}(b)\circ g)(i;y)$ where $g(i;y)=(i;f_i\,y)$.

g is an order isomorphism because $g(i;y_0)\geqslant g(i;y_1)\Leftrightarrow f_i\,y_0\geqslant f_i\,y_1\Leftrightarrow y_0\geqslant y_1$ and $i_0>i_1\Rightarrow g(i_0;y_0)>g(i_1;y_1)$. (Injectivity and surjectivity are obvious.) □

Let now S be an ordinal indexed family of ordinal indexed families of functions taking an ordinal number of arguments.

Lemma 3.105. $\operatorname{uncurry}(\operatorname{uncurry}\circ S)\sim\operatorname{uncurry}(\operatorname{uncurry}S)$.

Proof. $\operatorname{uncurry}\circ S=\lambda i\in S:\operatorname{uncurry}(S_i)$;

$\operatorname{uncurry}(\operatorname{uncurry}\circ S)(i;(x;y))=(\operatorname{uncurry}S_i)(x;y)=(S_i\,x)y$;

$(\operatorname{uncurry}(\operatorname{uncurry}S))((i;x);y)=((\operatorname{uncurry}S)(i;x))y=(S_i\,x)y$.