Lemma 3.94. $\left\{\operatorname{curry}(f)\circ\bigoplus(\operatorname{arity}\circ F)\mid f\in\operatorname{GR}\prod^{(\operatorname{ord})}F\right\}=\prod(\operatorname{GR}\circ F)$.

Proof. First $\operatorname{GR}\prod^{(\operatorname{ord})}F=\{\operatorname{uncurry}(z)\circ(\bigoplus(\operatorname{dom}\circ z))^{-1}\mid z\in\prod(\operatorname{GR}\circ F)\}$, that is $\left\{f\mid f\in\operatorname{GR}\prod^{(\operatorname{ord})}F\right\}=\{\operatorname{uncurry}(z)\circ(\bigoplus(\operatorname{arity}\circ F))^{-1}\mid z\in\prod(\operatorname{GR}\circ F)\}$.

Since $\bigoplus(\operatorname{arity}\circ F)$ is a bijection, we have $\left\{f\circ\bigoplus(\operatorname{arity}\circ F)\mid f\in\operatorname{GR}\prod^{(\operatorname{ord})}F\right\}=\{\operatorname{uncurry}(z)\mid z\in\prod(\operatorname{GR}\circ F)\}$ what is equivalent to $\left\{\operatorname{curry}(f)\circ\bigoplus(\operatorname{arity}\circ F)\mid f\in\operatorname{GR}\prod^{(\operatorname{ord})}F\right\}=\{z\mid z\in\prod(\operatorname{GR}\circ F)\}$ that is $\left\{\operatorname{curry}(f)\circ\bigoplus(\operatorname{arity}\circ F)\mid f\in\operatorname{GR}\prod^{(\operatorname{ord})}F\right\}=\prod(\operatorname{GR}\circ F)$. □

Lemma 3.95. $\{\bigcup\operatorname{im}P\mid P\in\coprod_{i\in\operatorname{dom}F}\mho^{\{i\}\times\operatorname{arity}F_i}\wedge\operatorname{curry}(\bigcup\operatorname{im}P)\in\prod(\operatorname{GR}\circ F)\}=\left\{L\in\mho^{\coprod_{i\in\operatorname{dom}F}\operatorname{arity}F_i}\mid\operatorname{curry}(L)\in\prod(\operatorname{GR}\circ F)\right\}$.

Proof. Let $L'\in\left\{L\in\mho^{\coprod_{i\in X}\operatorname{arity}F_i}\mid\operatorname{curry}(L)\in\prod(\operatorname{GR}\circ F)\right\}$. Then $L'\in\mho^{\coprod_{i\in\operatorname{dom}F}\operatorname{arity}F_i}$ and $\operatorname{curry}(L')\in\prod(\operatorname{GR}\circ F)$.

Let $P=\lambda i\in\operatorname{dom}F\colon L'|_{\{i\}\times\operatorname{arity}F_i}$. Then $P\in\coprod_{i\in\operatorname{dom}F}\mho^{\{i\}\times\operatorname{arity}F_i}$ and $\bigcup\operatorname{im}P=L'$. So $L'\in\{\bigcup\operatorname{im}P\mid P\in\coprod_{i\in\operatorname{dom}F}\mho^{\{i\}\times\operatorname{arity}F_i}\wedge\operatorname{curry}(\bigcup\operatorname{im}P)\in\prod(\operatorname{GR}\circ F)\}$.

Let now $L'\in\{\bigcup\operatorname{im}P\mid P\in\coprod_{i\in\operatorname{dom}F}\mho^{\{i\}\times\operatorname{arity}F_i}\wedge\operatorname{curry}(\bigcup\operatorname{im}P)\in\prod(\operatorname{GR}\circ F)\}$. Then there exists $P\in\coprod_{i\in\operatorname{dom}F}\mho^{\{i\}\times\operatorname{arity}F_i}$ such that $L'=\bigcup\operatorname{im}P$ and $\operatorname{curry}(L')\in\prod(\operatorname{GR}\circ F)$. Evidently $L'\in\mho^{\coprod_{i\in\operatorname{dom}F}\operatorname{arity}F_i}$. So $L'\in\{\bigcup\operatorname{im}P\mid P\in\coprod_{i\in\operatorname{dom}F}\mho^{\{i\}\times\operatorname{arity}F_i}\wedge\operatorname{curry}(\bigcup\operatorname{im}P)\in\prod(\operatorname{GR}\circ F)\}$. □

Lemma 3.96. $\left\{f\circ\bigoplus(\operatorname{arity}\circ F)\mid f\in\operatorname{GR}\prod^{(\operatorname{ord})}F\right\}=\{\bigcup\operatorname{im}P\mid P\in\prod_{i\in\operatorname{dom}F}F_i'\}$.

Proof. $L\in\{\bigcup\operatorname{im}P\mid P\in\prod_{i\in\operatorname{dom}F}F_i'\}\Leftrightarrow L\in\{\bigcup\operatorname{im}P\mid P\in\coprod_{i\in\operatorname{dom}F}\mho^{\{i\}\times\operatorname{arity}F_i}\wedge\operatorname{curry}(\bigcup\operatorname{im}P)\in\prod(\operatorname{GR}\circ F)\}\Leftrightarrow L\in\mho^{\coprod_{i\in\operatorname{dom}F}\operatorname{arity}F_i}\wedge\operatorname{curry}(L)\in\prod(\operatorname{GR}\circ F)\Leftrightarrow L\in\mho^{\coprod_{i\in\operatorname{dom}F}\operatorname{arity}F_i}\wedge\operatorname{curry}(L)\in\left\{\operatorname{curry}(f)\circ\bigoplus(\operatorname{arity}\circ F)\mid f\in\operatorname{GR}\prod^{(\operatorname{ord})}F\right\}\Leftrightarrow$ (because $\bigoplus(\operatorname{arity}\circ F)$ is a bijection)$\Leftrightarrow\operatorname{curry}(L)\circ(\bigoplus(\operatorname{arity}\circ F))^{-1}\in\left\{\operatorname{curry}(f)\mid f\in\operatorname{GR}\prod^{(\operatorname{ord})}F\right\}\Leftrightarrow L\circ(\bigoplus(\operatorname{arity}\circ F))^{-1}\in\left\{f\mid f\in\operatorname{GR}\prod^{(\operatorname{ord})}F\right\}\Leftrightarrow$ (because $\bigoplus(\operatorname{arity}\circ F)$ is a bijection)$\Leftrightarrow L\in\left\{f\circ\bigoplus(\operatorname{arity}\circ F)\mid f\in\operatorname{GR}\prod^{(\operatorname{ord})}F\right\}$. □

Theorem 3.97.

$$\operatorname{GR}\prod^{(\operatorname{ord})}F=\left\{\left(\bigcup\operatorname{im}P\right)\circ\left(\bigoplus(\operatorname{arity}\circ F)\right)^{-1}\mid P\in\prod_{i\in\operatorname{dom}F}F_i'\right\}.$$

Proof. From the lemma, because $\bigoplus(\operatorname{arity}\circ F)$ is a bijection. □

Theorem 3.98.

$$\operatorname{GR}\prod^{(\operatorname{ord})}F=\left\{\bigcup_{i\in\operatorname{dom}F}\left(P_i\circ\left(\bigoplus(\operatorname{arity}\circ F)\right)^{-1}\right)\mid P\in\prod_{i\in\operatorname{dom}F}F_i'\right\}.$$

Proof. From the previous theorem. □

Theorem 3.99.

$$\operatorname{GR}\prod^{(\operatorname{ord})}F=\left\{\bigcup\operatorname{im}P\mid P\in\prod_{i\in\operatorname{dom}F}\{f\circ\left(\bigoplus(\operatorname{arity}\circ F)\right)^{-1}\mid f\in F_i'\}\right\}.$$