Proof. $\operatorname{dom}\operatorname{concat}z = \sum_{i\in\operatorname{dom}F}\operatorname{dom}z_i = \sum_{i\in\operatorname{dom}F}\operatorname{arity}F_i = \sum(\operatorname{arity}\circ F)$. □

3.8.4.5 Definition with composition for every multiplier

$q(F)_i \overset{\text{def}}{=} (\operatorname{curry}(\bigoplus(\operatorname{arity}\circ F)))i$.

Theorem 3.87. $\operatorname{GR}\prod^{(\mathrm{ord})}F = \left\{L\in\mho^{\sum(\operatorname{arity}\circ F)} \mid \forall i\in\operatorname{dom}F: L\circ q(F)_i\in\operatorname{GR}F_i\right\}$.

Proof. $\operatorname{GR}\prod^{(\mathrm{ord})}F = \{\operatorname{concat}z \mid z\in\prod(\operatorname{GR}\circ F)\}$;

$\operatorname{GR}\prod^{(\mathrm{ord})}F = \left\{\operatorname{uncurry}(z)\circ(\bigoplus(\operatorname{arity}\circ F))^{-1} \mid z\in\prod_{i\in\operatorname{dom}F}\mho^{\operatorname{arity}F_i}, \forall i\in\operatorname{dom}F: z(i)\in\operatorname{GR}F_i\right\}$.

Let $L = \operatorname{uncurry}(z)$. Then $z = \operatorname{curry}(L)$.

$\operatorname{GR}\prod^{(\mathrm{ord})}F = \left\{L\circ(\bigoplus(\operatorname{arity}\circ F))^{-1} \mid \operatorname{curry}(L)\in\prod_{i\in\operatorname{dom}F}\mho^{\operatorname{arity}F_i}, \forall i\in\operatorname{dom}F: \operatorname{curry}(L)i\in\operatorname{GR}F_i\right\}$;

$\operatorname{GR}\prod^{(\mathrm{ord})}F = \left\{L\circ(\bigoplus(\operatorname{arity}\circ F))^{-1} \mid L\in\mho^{\coprod_{i\in\operatorname{dom}F}\operatorname{arity}F_i}, \forall i\in\operatorname{dom}F: \operatorname{curry}(L)i\in\operatorname{GR}F_i\right\}$;

$\operatorname{GR}\prod^{(\mathrm{ord})}F = \left\{L\in\mho^{\sum(\operatorname{arity}\circ F)} \mid \forall i\in\operatorname{dom}F: \operatorname{curry}(L\circ\bigoplus(\operatorname{arity}\circ F))i\in\operatorname{GR}F_i\right\}$;

$(\operatorname{curry}(L\circ\bigoplus(\operatorname{arity}\circ F))i)x = L((\operatorname{curry}(\bigoplus(\operatorname{arity}\circ F))i)x) = L(q(F)_i x) = (L\circ q(F)_i)x$;

$\operatorname{curry}(L\circ\bigoplus(\operatorname{arity}\circ F))i = L\circ q(F)_i$;

$\operatorname{GR}\prod^{(\mathrm{ord})}F = \left\{L\in\mho^{\sum(\operatorname{arity}\circ F)} \mid \forall i\in\operatorname{dom}F: L\circ q(F)_i\in\operatorname{GR}F_i\right\}$. □

Corollary 3.88. $\operatorname{GR}\prod^{(\mathrm{ord})}F = \left\{L\in(\bigcup\operatorname{im}(\operatorname{GR}\circ F))^{\sum(\operatorname{arity}\circ F)} \mid \forall i\in\operatorname{dom}F: L\circ q(F)_i\in\operatorname{GR}F_i\right\}$.

Corollary 3.89. $\operatorname{GR}\prod^{(\mathrm{ord})}F$ is small if F is small.

3.8.4.6 Definition with shifting arguments

Let $F_i' = \{L\circ\Pr_1|_{\{i\}\times\operatorname{arity}F_i} \mid L\in\operatorname{GR}F_i\}$.

Proposition 3.90. $F_i' = \{L\circ\Pr_1|_{\{i\}\times\mho} \mid L\in\operatorname{GR}F_i\}$.

Proof. If $L\in\operatorname{GR}F_i$ then $\operatorname{dom}L = \operatorname{arity}F_i$. Thus

$$L\circ\Pr_1|_{\{i\}\times\operatorname{arity}F_i} = L\circ\Pr_1|_{\{i\}\times\operatorname{dom}L} = L\circ\Pr_1|_{\{i\}\times\mho}.$$ □

Proposition 3.91. F_i' is an $(\{i\}\times\operatorname{arity}F_i)$-ary relation.

Proof. We need to prove that $\operatorname{dom}(L\circ\Pr_1|_{\{i\}\times\operatorname{arity}F_i}) = \{i\}\times\operatorname{arity}F_i$ for $L\in\operatorname{GR}F_i$, but that's obvious. □

Obvious 3.92. $\coprod(\operatorname{arity}\circ F) = \bigcup_{i\in\operatorname{dom}F}(\{i\}\times\operatorname{arity}F_i) = \bigcup_{i\in\operatorname{dom}F}\operatorname{dom}F_i'$.

Lemma 3.93. $P\in\prod_{i\in\operatorname{dom}F}F_i' \Leftrightarrow \operatorname{curry}(\bigcup\operatorname{im}P)\in\prod(\operatorname{GR}\circ F)$ for a $\operatorname{dom}F$ indexed family P where $P_i\in\mho^{\{i\}\times\operatorname{arity}F_i}$ for every $i\in\operatorname{dom}F$, that is for $P\in\coprod_{i\in\operatorname{dom}F}\mho^{\{i\}\times\operatorname{arity}F_i}$.

Proof. For every $P\in\coprod_{i\in\operatorname{dom}F}\mho^{\{i\}\times\operatorname{arity}F_i}$ we have:

$P\in\prod_{i\in\operatorname{dom}F}F_i' \Leftrightarrow P\in\{z\in\mho^{\operatorname{dom}F} \mid \forall i\in\operatorname{dom}F: z(i)\in F_i'\} \Leftrightarrow P\in\mho^{\operatorname{dom}F}\wedge\forall i\in\operatorname{dom}F: P(i)\in F_i' \Leftrightarrow P\in\mho^{\operatorname{dom}F}\wedge\forall i\in\operatorname{dom}F\exists L\in\operatorname{GR}F_i: P_i = L\circ(\Pr_1|_{\{i\}\times\mho}) \Leftrightarrow P\in\mho^{\operatorname{dom}F}\wedge\forall i\in\operatorname{dom}F\exists L\in\operatorname{GR}F_i: \left(P_i\in\mho^{\{i\}\times\operatorname{arity}F_i}\wedge\forall x\in\operatorname{arity}F_i: P_i(i;x) = Lx\right) \Leftrightarrow P\in\mho^{\operatorname{dom}F}\wedge\forall i\in\operatorname{dom}F\exists L\in\operatorname{GR}F_i: \left(P_i\in\mho^{\{i\}\times\operatorname{arity}F_i}\wedge\operatorname{curry}(P_i)i = L\right) \Leftrightarrow P\in\mho^{\operatorname{dom}F}\wedge\forall i\in\operatorname{dom}F: \left(P_i\in\mho^{\{i\}\times\operatorname{arity}F_i}\wedge\operatorname{curry}(P_i)i\in\operatorname{GR}F_i\right) \Leftrightarrow \forall i\in\operatorname{dom}F\exists Q_i\in(\mho^{\operatorname{arity}F_i})^{\{i\}}: (P_i = \operatorname{uncurry}(Q_i)\wedge(Q_i)i\in\mho^{\operatorname{arity}F_i}\wedge Q_i i\in\operatorname{GR}F_i) \Leftrightarrow \forall i\in\operatorname{dom}F\exists Q_i\in(\mho^{\operatorname{arity}F_i})^{\{i\}}\left(P_i = \operatorname{uncurry}(Q_i)\wedge\left(\bigcup_{i\in\operatorname{dom}F}Q_i\right)i\in\operatorname{GR}F_i\right) \Leftrightarrow \forall i\in\operatorname{dom}F\exists Q_i\in(\mho^{\operatorname{arity}F_i})^{\{i\}}\left(P_i = \operatorname{uncurry}(Q_i)\wedge\bigcup_{i\in\operatorname{dom}F}Q_i\in\prod(\operatorname{GR}\circ F)\right) \Leftrightarrow \forall i\in\operatorname{dom}F: \bigcup_{i\in\operatorname{dom}F}\operatorname{curry}(P_i)\in\prod(\operatorname{GR}\circ F) \Leftrightarrow \operatorname{curry}\left(\bigcup_{i\in\operatorname{dom}F}P_i\right)\in\prod(\operatorname{GR}\circ F) \Leftrightarrow \operatorname{curry}(\bigcup\operatorname{im}P)\in\prod(\operatorname{GR}\circ F)$. □