3.8.4 Ordinated product

3.8.4.1 Introduction

Ordinated product defined below is a variation of Cartesian product, but is associative unlike Cartesian product. However, ordinated product unlike Cartesian product is defined not for arbitrary sets, but only for relations having ordinal numbers of arguments.

Let F indexed by an ordinal number be a small family of anchored relations.

3.8.4.2 Concatenation

Definition 3.81. Let z be an indexed by an ordinal number family of functions each taking an ordinal number of arguments. The *concatenation* of z is

$$\operatorname{concat} z = \operatorname{uncurry}(z) \circ \left(\bigoplus (\operatorname{dom} \circ z)\right)^{-1}.$$

Obvious 3.82. If z is a finite family of finitary typles, it is concatenation of $\operatorname{dom} z$ tuples in the usual sense (as it is commonly used in computer science).

Proposition 3.83. If $z \in \prod (\mathrm{GR} \circ F)$ then $\operatorname{concat} z = \operatorname{uncurry}(z) \circ (\bigoplus (\operatorname{arity} \circ F))^{-1}$.

Proof. If $z \in \prod (\mathrm{GR} \circ F)$ then $\operatorname{dom} z(i) = \operatorname{dom} (\mathrm{GR} \circ F)_i = \operatorname{dom} F_i = \operatorname{arity} F_i$ for every $i \in \operatorname{dom} F$. Thus $\operatorname{dom} \circ z = \operatorname{arity} \circ F$. □

Proposition 3.84. $\operatorname{dom} \operatorname{concat} z = \sum_{i \in \operatorname{dom} z} \operatorname{dom} z_i$.

Proof. Because $\operatorname{dom} (\bigoplus (\operatorname{dom} \circ z))^{-1} = \sum_{i \in \operatorname{dom} F} \operatorname{dom} z_i$, it is enough to prove that

$$\operatorname{dom} \operatorname{uncurry}(z) = \operatorname{dom} \bigoplus (\operatorname{dom} \circ z).$$

Really,

$$\operatorname{dom} \bigoplus (\operatorname{dom} \circ z) = \{(i; x) \mid i \in \operatorname{dom} (\operatorname{dom} \circ z), x \in \operatorname{dom} z_i\} = \{(i; x) \mid i \in \operatorname{dom} z, x \in \operatorname{dom} z_i\} = \coprod z$$

and $\operatorname{dom} \operatorname{uncurry}(z) = \coprod_{i \in X} z_i = \coprod z$. □

3.8.4.3 Finite example

If F is a finite family (indexed by a natural number $\operatorname{dom} F$) of anchored finitary relations, then by definition $\mathrm{GR} \prod^{(\mathrm{ord})} F = \{[\![a_{0,0}; \ldots; a_{0,\operatorname{arity} F_0 - 1}; \ldots; a_{\operatorname{dom} F - 1, 0}; \ldots; a_{\operatorname{dom} F - 1, \operatorname{arity} F_{\operatorname{dom} F - 1} - 1}]\!] \mid [\![a_{0,0}; \ldots; a_{0,\operatorname{arity} F_0 - 1}]\!] \in \mathrm{GR} F_0 \wedge \ldots \wedge [\![a_{\operatorname{dom} F - 1, \operatorname{arity} F_{\operatorname{dom} F - 1} - 1}]\!] \in \mathrm{GR} F_{\operatorname{dom} F - 1}\}$ and

$$\operatorname{arity} \prod^{(\mathrm{ord})} F = \operatorname{arity} F_0 + \ldots + \operatorname{arity} F_{\operatorname{dom} F - 1}.$$

The above formula can be shortened to

$$\mathrm{GR} \prod^{(\mathrm{ord})} F = \left\{\operatorname{concat} z \mid z \in \prod (\mathrm{GR} \circ F)\right\}.$$

3.8.4.4 The definition

Definition 3.85. The anchored relation (which I call *ordinated product*) $\prod^{(\mathrm{ord})} F$ is defined by the formulas:

$$\operatorname{arity} \prod^{(\mathrm{ord})} F = \sum (\operatorname{arity} \circ F);$$
$$\mathrm{GR} \prod^{(\mathrm{ord})} F = \left\{\operatorname{concat} z \mid z \in \prod (\mathrm{GR} \circ F)\right\}.$$

Proposition 3.86. $\prod^{(\mathrm{ord})} F$ is a properly defined anchored relation.