Obvious 3.74.

1. $\operatorname{uncurry}(\operatorname{curry}(f)) = f$ for every $f \in Z^{X \times Y}$.
2. $\operatorname{curry}(\operatorname{uncurry}(f)) = f$ for every $f \in (Z^Y)^X$.

Currying and uncurrying with a dependent variable

Let X, Z be sets and Y be a function with the domain X. (Vaguely saying, Y is a variable dependent on X.)

The disjoint union $\coprod Y = \bigcup \{\{i\} \times Y_i \mid i \in \operatorname{dom} Y\} = \{(i; x) \mid i \in \operatorname{dom} Y, x \in Y_i\}$.

We will consider variables $x \in X$ and $y \in Y_x$.

Let a function $f \in Z^{\coprod_{i \in X} Y_i}$ (or equivalently $f \in Z^{\coprod Y}$). Then $\operatorname{curry}(f) \in \prod_{i \in X} Z^{Y_i}$ is the function defined by the formula $(\operatorname{curry}(f)x)\, y = f(x; y)$.

Let now $f \in \prod_{i \in X} Z^{Y_i}$. Then $\operatorname{uncurry}(f) \in Z^{\coprod_{i \in X} Y_i}$ is the function defined by the formula $\operatorname{uncurry}(f)(x; y) = (f x)\, y$.

Obvious 3.75.

1. $\operatorname{uncurry}(\operatorname{curry}(f)) = f$ for every $f \in Z^{\coprod_{i \in X} Y_i}$.
2. $\operatorname{curry}(\operatorname{uncurry}(f)) = f$ for every $f \in \prod_{i \in X} Z^{Y_i}$.

3.8.2.2 Functions with ordinal numbers of arguments

Let Ord be the set of small ordinal numbers.

If X and Y are sets and n is an ordinal number, the set of functions taking n arguments on the set X and returning a value in Y is Y^{X^n}.

The set of all small functions taking ordinal numbers of arguments is $Y^{\bigcup_{n \in \mathrm{Ord}} X^n}$.

I will denote $\operatorname{OrdVar}(X) = \mho^{\bigcup_{n \in \mathrm{Ord}} X^n}$ and call it *ordinal variadic*. ("Var" in this notation is taken from the word *variadic* in the collocation *variadic function* used in computer science.)

3.8.3 On sums of ordinals

Let a be an ordinal-indexed family of ordinals.

Proposition 3.76. $\coprod a$ with lexicographic order is a well-ordered set.

Proof. Let S be non-empty subset of $\coprod a$.

Take $i_0 = \min \Pr_0 S$ and $x_0 = \min \{\Pr_1 y \mid y \in S, y(0) = i_0\}$ (these exist by properties of ordinals). Then $(i_0; x_0)$ is the least element of S. □

Definition 3.77. $\sum a$ is the unique ordinal order-isomorphic to $\coprod a$. [TODO: For finite ordinals it is just a sum of natural numbers.]

This ordinal exists and is unique because our set is well-ordered.

Remark 3.78. An infinite sum of ordinals is not customary defined.

The *structured sum* $\bigoplus a$ of a is an order isomorphism from lexicographically ordered set $\coprod a$ into $\sum a$.

There exists (for a given a) exactly one structured sum, by properties of well-ordered sets.

Obvious 3.79. $\sum a = \operatorname{im} \bigoplus a$.

Theorem 3.80. $(\bigoplus a)(n; x) = \sum_{i \in n} a_i + x$.

Proof. We need to prove that it is an order isomorphism. Let's prove it is an injection that is $m > n \Rightarrow \sum_{i \in m} a_i + x > \sum_{i \in n} a_i + x$ and $y > x \Rightarrow \sum_{i \in n} a_i + y > \sum_{i \in n} a_i + x$.

Really, if $m > n$ then $\sum_{i \in m} a_i + x \geqslant \sum_{i \in n+1} a_i + x > \sum_{i \in n} a_i + x$. The second formula is true by properties of ordinals.

Let's prove that it is a surjection. Let $r \in \sum a$. There exist $n \in \operatorname{dom} a$ and $x \in a_n$ such that $r = (\bigoplus a)(n; x)$. Thus $r = (\bigoplus a)(n; 0) + x = \sum_{i \in n} a_i + x$ because $(\bigoplus a)(n; 0) = \sum_{i \in n} a_i$ since $(n; 0)$ has $\sum_{i \in n} a_i$ predecessors. □