Definition 3.50. *Transitive* (endo)morphism of a precategory is such a morphism f that $f = f \circ f$.

Theorem 3.51. The following conditions are equivalent for a morphism f of a dagger precategory:

1. f is symmetric and transitive.
2. $f = f^\dagger \circ f$.

Proof.

(1)⇒(2). If f is symmetric and transitive then $f^\dagger \circ f = f \circ f = f$.

(2)⇒(1). $f^\dagger = (f^\dagger \circ f)^\dagger = f^\dagger \circ f^{\dagger\dagger} = f^\dagger \circ f = f$, so f is symmetric. $f = f^\dagger \circ f = f \circ f$, so f is transitive. □

3.5.2.1 Some special classes of morphisms

Definition 3.52. For a partially ordered dagger category I will call *monovalued* morphism such a morphism f that $f \circ f^\dagger \sqsubseteq 1_{\mathrm{Dst}\, f}$.

Definition 3.53. For a partially ordered dagger category I will call *entirely defined* morphism such a morphism f that $f^\dagger \circ f \sqsupseteq 1_{\mathrm{Src}\, f}$.

Definition 3.54. For a partially ordered dagger category I will call *injective* morphism such a morphism f that $f^\dagger \circ f \sqsubseteq 1_{\mathrm{Src}\, f}$.

Definition 3.55. For a partially ordered dagger category I will call *surjective* morphism such a morphism f that $f \circ f^\dagger \sqsupseteq 1_{\mathrm{Dst}\, f}$.

Remark 3.56. It is easy to show that this is a generalization of monovalued, entirely defined, injective, and surjective functions as morphisms of the category Rel.

Obvious 3.57. "Injective morphism" is a dual of "monovalued morphism" and "surjective morphism" is a dual of "entirely defined morphism".

Definition 3.58. For a given partially ordered dagger category C the *category of monovalued* (*entirely defined*, *injective*, *surjective*) morphisms of C is the category with the same set of objects as of C and the set of morphisms being the set of monovalued (entirely defined, injective, surjective) morphisms of C with the composition of morphisms the same as in C.

We need to prove that these are really categories, that is that composition of monovalued (entirely defined, injective, surjective) morphisms is monovalued (entirely defined, injective, surjective) and that identity morphisms are monovalued, entirely defined, injective, and surjective.

Proof. We will prove only for monovalued morphisms and entirely defined morphisms, as injective and surjective morphisms are their duals.

Monovalued. Let f and g be monovalued morphisms, $\mathrm{Dst}\, f = \mathrm{Src}\, g$. $(g \circ f) \circ (g \circ f)^\dagger = g \circ f \circ f^\dagger \circ g^\dagger \sqsubseteq g \circ 1_{\mathrm{Dst}\, f} \circ g^\dagger = g \circ 1_{\mathrm{Src}\, g} \circ g^\dagger = g \circ g^\dagger \sqsubseteq 1_{\mathrm{Dst}\, g} = 1_{\mathrm{Dst}(g \circ f)}$. So $g \circ f$ is monovalued.

That identity morphisms are monovalued follows from the following: $1_A \circ (1_A)^\dagger = 1_A \circ 1_A = 1_A = 1_{\mathrm{Dst}\, 1_A} \sqsubseteq 1_{\mathrm{Dst}\, 1_A}$.

Entirely defined. Let f and g be entirely defined morphisms, $\mathrm{Dst}\, f = \mathrm{Src}\, g$. $(g \circ f)^\dagger \circ (g \circ f) = f^\dagger \circ g^\dagger \circ g \circ f \sqsupseteq f^\dagger \circ 1_{\mathrm{Src}\, g} \circ f = f^\dagger \circ 1_{\mathrm{Dst}\, f} \circ f = f^\dagger \circ f \sqsupseteq 1_{\mathrm{Src}\, f} = 1_{\mathrm{Src}(g \circ f)}$. So $g \circ f$ is entirely defined.

That identity morphisms are entirely defined follows from the following:
$(1_A)^\dagger \circ 1_A = 1_A \circ 1_A = 1_A = 1_{\mathrm{Src}\, 1_A} \sqsupseteq 1_{\mathrm{Src}\, 1_A}$. □

Definition 3.59. I will call a *bijective* morphism a morphism which is entirely defined, monovalued, injective, and surjective.

Proposition 3.60. If a morphism is bijective then it is an isomorphism.