that is

$$\bigsqcup \{z \in \mathfrak{A} \mid z \sqsubseteq a \wedge z \sqcap b = 0^{\mathfrak{A}}\} = \bigsqcup \{z \in \operatorname{atoms} a \mid z \sqcap b = 0^{\mathfrak{A}}\} = \bigsqcup (\operatorname{atoms} a \setminus \operatorname{atoms} b). \qquad \square$$

3.5 Partially ordered categories

3.5.1 Definition

Definition 3.44. I will call a partially ordered (pre)category a (pre)category together with partial order $\sqsubseteq$ on each of its Mor-sets with the additional requirement that

$$f_1 \sqsubseteq f_2 \wedge g_1 \sqsubseteq g_2 \Rightarrow g_1 \circ f_1 \sqsubseteq g_2 \circ f_2$$

for every morphisms f_1, g_1, f_2, g_2 such that $\operatorname{Src} f_1 = \operatorname{Src} f_2 \wedge \operatorname{Dst} f_1 = \operatorname{Dst} f_2 = \operatorname{Src} g_1 = \operatorname{Src} g_2 \wedge \operatorname{Dst} g_1 = \operatorname{Dst} g_2$.

3.5.2 Dagger categories

Definition 3.45. I will call a *dagger precategory* a precategory together with an involutive contravariant identity-on-objects prefunctor $x \mapsto x^{\dagger}$.

In other words, a dagger precategory is a precategory equipped with a function $x \mapsto x^{\dagger}$ on its set of morphisms which reverses the source and the destination and is subject to the following identities for every morphisms f and g:

1. $f^{\dagger\dagger} = f$;
2. $(g \circ f)^{\dagger} = f^{\dagger} \circ g^{\dagger}$.

Definition 3.46. I will call a *dagger category* a category together with an involutive contravariant identity-on-objects functor $x \mapsto x^{\dagger}$.

In other words, a dagger category is a category equipped with a function $x \mapsto x^{\dagger}$ on its set of morphisms which reverses the source and the destination and is subject to the following identities for every morphisms f and g and object A:

1. $f^{\dagger\dagger} = f$;
2. $(g \circ f)^{\dagger} = f^{\dagger} \circ g^{\dagger}$;
3. $(1_A)^{\dagger} = 1_A$.

Theorem 3.47. If a category is a dagger precategory then it is a dagger category.

Proof. We need to prove only that $(1_A)^{\dagger} = 1_A$. Really,

$$(1_A)^{\dagger} = (1_A)^{\dagger} \circ 1_A = (1_A)^{\dagger} \circ (1_A)^{\dagger\dagger} = ((1_A)^{\dagger} \circ 1_A)^{\dagger} = (1_A)^{\dagger\dagger} = 1_A. \qquad \square$$

For a partially ordered dagger (pre)category I will additionally require (for every morphisms f and g with the same source and destination)

$$f^{\dagger} \sqsubseteq g^{\dagger} \Leftrightarrow f \sqsubseteq g.$$

An example of dagger category is the category Rel whose objects are sets and whose morphisms are binary relations between these sets with usual composition of binary relations and with $f^{\dagger} = f^{-1}$.

Definition 3.48. A morphism f of a dagger category is called *unitary* when it is an isomorphism and $f^{\dagger} = f^{-1}$.

Definition 3.49. *Symmetric* (endo)morphism of a dagger precategory is such a morphism f that $f = f^{\dagger}$.