2.

$$\begin{aligned} (a \sqcap b)^{*(Da)} &= \\ \bigsqcup \{c \in Da \mid c \sqcap a \sqcap b = 0\} &= \\ \bigsqcup \{c \in \mathfrak{A} \mid c \sqsubseteq a \wedge c \sqcap a \sqcap b = 0\} &= \\ \bigsqcup \{c \in \mathfrak{A} \mid c \sqsubseteq a \wedge c \sqcap b = 0\} &= \\ a \# b. & \end{aligned}$$

□

Proposition 3.42. $(a \sqcup b) \setminus^* b \sqsubseteq a$ for an arbitrary complete lattice.

Proof. $(a \sqcup b) \setminus^* b = \bigsqcap \{z \in \mathfrak{A} \mid a \sqcup b \sqsubseteq b \sqcup z\}$.

But $a \sqsubseteq z \Rightarrow a \sqcup b \sqsubseteq b \sqcup z$. So $\{z \in \mathfrak{A} \mid a \sqcup b \sqsubseteq b \sqcup z\} \supseteq \{z \in \mathfrak{A} \mid a \sqsubseteq z\}$.

Consequently, $(a \sqcup b) \setminus^* b \sqsubseteq \bigsqcap \{z \in \mathfrak{A} \mid a \sqsubseteq z\} = a$. □

3.4 Several equal ways to express pseudodifference

Theorem 3.43. For an atomistic co-brouwerian lattice $\mathfrak{A}$ and $a, b \in \mathfrak{A}$ the following expressions are always equal:

1. $a \setminus^* b = \bigsqcap \{z \in \mathfrak{A} \mid a \sqsubseteq b \sqcup z\}$ (quasidifference of a and b);
2. $a \# b = \bigsqcup \{z \in \mathfrak{A} \mid z \sqsubseteq a \wedge z \sqcap b = 0\}$ (second quasidifference of a and b);
3. $\bigsqcup (\operatorname{atoms} a \setminus \operatorname{atoms} b)$.

Proof. *Proof of (1)=(3):*

$$\begin{aligned} a \setminus^* b &= \\ \left(\bigsqcup \operatorname{atoms} a\right) \setminus^* b &= \quad \text{(theorem 2.123)} \\ \bigsqcup \{A \setminus^* b \mid A \in \operatorname{atoms} a\} &= \\ \bigsqcup \left\{ \left(\begin{cases} A & \text{if } A \notin \operatorname{atoms} b \\ 0 & \text{if } A \in \operatorname{atoms} b \end{cases} \right) \mid A \in \operatorname{atoms} a \right\} &= \\ \bigsqcup \{A \mid A \in \operatorname{atoms} a, A \notin \operatorname{atoms} b\} &= \\ \bigsqcup (\operatorname{atoms} a \setminus \operatorname{atoms} b). & \end{aligned}$$

Proof of (2)=(3):

$a \setminus^* b$ is defined because our lattice is co-brouwerian. Taking the above into account, we have

$$\begin{aligned} a \setminus^* b &= \\ \bigsqcup (\operatorname{atoms} a \setminus \operatorname{atoms} b) &= \\ \bigsqcup \{z \in \operatorname{atoms} a \mid z \sqcap b = 0^{\mathfrak{A}}\}. & \end{aligned}$$

So $\bigsqcup \{z \in \operatorname{atoms} a \mid z \sqcap b = 0^{\mathfrak{A}}\}$ is defined.

If $z \sqsubseteq a \wedge z \sqcap b = 0^{\mathfrak{A}}$ then $z' = \bigsqcup \{x \in \operatorname{atoms} z \mid x \sqcap b = 0^{\mathfrak{A}}\}$ is defined. z' is a lower bound for $\{z \in \operatorname{atoms} a \mid z \sqcap b = 0^{\mathfrak{A}}\}$.

Thus $z' \in \{z \in \mathfrak{A} \mid z \sqsubseteq a \wedge z \sqcap b = 0^{\mathfrak{A}}\}$ and so $\bigsqcup \{z \in \operatorname{atoms} a \mid z \sqcap b = 0^{\mathfrak{A}}\}$ is an upper bound of $\{z \in \mathfrak{A} \mid z \sqsubseteq a \wedge z \sqcap b = 0^{\mathfrak{A}}\}$.

If y is above every $z' \in \{z \in \mathfrak{A} \mid z \sqsubseteq a \wedge z \sqcap b = 0^{\mathfrak{A}}\}$ then y is above every $z \in \operatorname{atoms} a$ such that $z \sqcap b = 0^{\mathfrak{A}}$ and thus y is above $\bigsqcup \{z \in \operatorname{atoms} a \mid z \sqcap b = 0^{\mathfrak{A}}\}$.

Thus $\bigsqcup \{z \in \operatorname{atoms} a \mid z \sqcap b = 0^{\mathfrak{A}}\}$ is least upper bound of

$$\{z \in \mathfrak{A} \mid z \sqsubseteq a \wedge z \sqcap b = 0^{\mathfrak{A}}\},$$