Proposition 3.27. Let $\mathfrak{A}$ be a join-semilattice. $S \in \mathscr{P}\mathfrak{A}$ is a free star iff the least element (if it exists) is not in S and for every $X, Y \in \mathfrak{A}$

$$X \sqcup Y \in S \Leftrightarrow X \in S \vee Y \in S.$$

Proof.

- **⇒.** We need to prove only $X \sqcup Y \in S \Leftarrow X \in S \vee Y \in S$ what follows from that S is an upper set.
- **⇐.** We need to prove only that S is an upper set. Let $X \in S$ and $X \sqsubseteq Y \in \mathfrak{A}$. Then $X \in S \Rightarrow X \in S \vee Y \in S \Leftrightarrow X \sqcup Y \in S \Rightarrow Y \in S$. So S is an upper set. □

3.2.1 Starrish posets

Definition 3.28. I will call a poset *starrish* when the full star $\star a$ is a free star for every element a of this poset.

Proposition 3.29. Every distributive lattice is starrish.

Proof. Let $\mathfrak{A}$ be a distributive lattice, $a \in \mathfrak{A}$. Obviously $0 \notin \star a$ (if 0 exists); obviously $\star a$ is an upper set. If $x \sqcup y \in \star a$, then $(x \sqcup y) \sqcap a$ is non-least that is $(x \sqcap a) \sqcup (y \sqcap a)$ is non-least what is equivalent to $x \sqcap a$ or $y \sqcap a$ being non-least that is $x \in \star a \vee y \in \star a$. □

Theorem 3.30. If $\mathfrak{A}$ is a starrish join-semilattice lattice then

$$\operatorname{atoms}(a \sqcup b) = \operatorname{atoms} a \cup \operatorname{atoms} b$$

for every $a, b \in \mathfrak{A}$.

Proof. For every atom c we have: $c \in \operatorname{atoms}(a \sqcup b) \Leftrightarrow c \not\asymp a \sqcup b \Leftrightarrow a \sqcup b \in \star c \Leftrightarrow a \in \star c \vee b \in \star c \Leftrightarrow c \not\asymp a \vee c \not\asymp b \Leftrightarrow c \in \operatorname{atoms} a \vee c \in \operatorname{atoms} b$. □

3.3 Quasidifference and Quasicomplement

I've got quasidifference and quasicomplement (and dual quasicomplement) replacing max and min in the definition of pseudodifference and pseudocomplement (and dual pseudocomplement) with $\bigsqcup$ and $\bigsqcap$. Thus quasidifference and (dual) quasicomplement are generalizations of their pseudo-counterparts.

Remark 3.31. *Pseudocomplements* and *pseudodifferences* are standard terminology. *Quasi-* counterparts are my neologisms.

Definition 3.32. Let $\mathfrak{A}$ be a poset, $a \in \mathfrak{A}$. *Quasicomplement* of a is

$$a^* = \bigsqcup \{c \in \mathfrak{A} \mid c \asymp a\}.$$

Definition 3.33. Let $\mathfrak{A}$ be a poset, $a \in \mathfrak{A}$. *Dual quasicomplement* of a is

$$a^+ = \bigsqcap \{c \in \mathfrak{A} \mid c \equiv a\}.$$

I will denote quasicomplement and dual quasicomplement for a specific poset $\mathfrak{A}$ as $a^{*(\mathfrak{A})}$ and $a^{+(\mathfrak{A})}$.

Definition 3.34. Let $a, b \in \mathfrak{A}$ where $\mathfrak{A}$ is a distributive lattice. *Quasidifference* of a and b is

$$a \setminus^* b = \bigsqcap \{z \in \mathfrak{A} \mid a \sqsubseteq b \sqcup z\}.$$

Definition 3.35. Let $a, b \in \mathfrak{A}$ where $\mathfrak{A}$ is a distributive lattice. *Second quasidifference* of a and b is

$$a \# b = \bigsqcup \{z \in \mathfrak{A} \mid z \sqsubseteq a \wedge z \asymp b\}.$$