Proof.

(1)⇔(2)⇔(3)⇔(4)⇔(5)⇔(6). By the above theorem.

(8)⇒(4). Let property (8) hold. Let $a \sqsubset b$. Then it exists element $c \sqsubseteq b$ such that $c \neq 0$ and $c \sqcap a = 0$. But $c \sqcap b \neq 0$. So $\star a \neq \star b$.

(2)⇒(7). Let property (2) hold. Let $a \not\sqsubseteq b$. Then $\star a \not\sqsubseteq \star b$ that is it there exists $c \in \star a$ such that $c \notin \star b$, in other words $c \sqcap a \neq 0$ and $c \sqcap b = 0$. Let $d = c \sqcap a$. Then $d \sqsubseteq a$ and $d \neq 0$ and $d \sqcap b = 0$. So disjunction property of Wallman holds.

(7)⇒(8). Obvious.

(8)⇒(7). Let $b \not\sqsubseteq a$. Then $a \sqcap b \sqsubset b$ that is $a' \sqsubset b$ where $a' = a \sqcap b$. Consequently $\exists c \in \mathfrak{A} \setminus \{0\}$: $(c \asymp a' \wedge c \sqsubseteq b)$. We have $c \sqcap a = c \sqcap b \sqcap a = c \sqcap a'$. So $c \sqsubseteq b$ and $c \sqcap a = 0$. Thus Wallman's disjunction property holds. □

Proposition 3.15. Every boolean lattice is separable.

Proof. Let $a, b \in \mathfrak{A}$ where $\mathfrak{A}$ is a boolean lattice an $a \neq b$. Then $a \sqcap \bar{b} \neq 0$ or $\bar{a} \sqcap b \neq 0$ because otherwise $a \sqcap \bar{b} = 0$ and $a \sqcup \bar{b} = 1$ and thus $a = b$. Without loss of generality assume $a \sqcap \bar{b} \neq 0$. Then $a \sqcap c \neq 0$ and $b \sqcap c = 0$ for $c = a \sqcap \bar{b} \neq 0$. □

3.1.3 Atomically Separable Lattices

Proposition 3.16. "atoms" is a straight monotone map (for any meet-semilattice).

Proof. Monotonicity is obvious. The rest follows from the formula

$$\operatorname{atoms}(a \sqcap b) = \operatorname{atoms} a \cap \operatorname{atoms} b$$

(the corollary 2.87). □

Definition 3.17. I will call *atomically separable* such a poset that "atoms" is an injection.

Proposition 3.18. $\forall a, b \in \mathfrak{A}: (a \sqsubset b \Rightarrow \operatorname{atoms} a \subset \operatorname{atoms} b)$ iff $\mathfrak{A}$ is atomically separable for a poset $\mathfrak{A}$.

Proof.

⇐. Obvious.

⇒. Let $a \neq b$ for example $a \not\sqsubseteq b$. Then $a \sqcap b \sqsubset a$; $\operatorname{atoms} a \supset \operatorname{atoms}(a \sqcap b) = \operatorname{atoms} a \cap \operatorname{atoms} b$ and thus $\operatorname{atoms} a \neq \operatorname{atoms} b$. □

Proposition 3.19. Any atomistic poset is atomically separable.

Proof. We need to prove that $\operatorname{atoms} a = \operatorname{atoms} b \Rightarrow a = b$. But it is obvious because

$$a = \bigsqcup \operatorname{atoms} a \quad \text{and} \quad b = \bigsqcup \operatorname{atoms} b.$$ □

Theorem 3.20. If a lattice with least element is atomic and separable then it is atomistic.

Proof. Suppose the contrary that is $a \sqsupset \bigsqcup \operatorname{atoms} a$. Then, because our lattice is separable, there exists $c \in \mathfrak{A}$ such that $c \sqcap a \neq 0$ and $c \sqcap \bigsqcup \operatorname{atoms} a = 0$. There exists atom $d \sqsubseteq c$ such that $d \sqsubseteq c \sqcap a$. $d \sqcap \bigsqcup \operatorname{atoms} a \sqsubseteq c \sqcap \bigsqcup \operatorname{atoms} a = 0$. But $d \in \operatorname{atoms} a$. Contradiction. □

Theorem 3.21. Let $\mathfrak{A}$ be an atomic meet-semilattice with least element. Then the following statements are equivalent:

1. $\mathfrak{A}$ is separable.
2. $\mathfrak{A}$ is atomically separable.
3. $\mathfrak{A}$ conforms to Wallman's disjunction property.