Chapter 3

More on order theory

3.1 Straight maps and separation subsets

3.1.1 Straight maps

Definition 3.1. Let f be a monotone map from a meet-semilattice $\mathfrak{A}$ to some poset $\mathfrak{B}$. I call f a *straight* map when

$$\forall a, b \in \mathfrak{A}: (fa \sqsubseteq fb \Rightarrow fa = f(a \sqcap b)).$$

Proposition 3.2. The following statements are equivalent for a monotone map f:

1. f is a straight map.
2. $\forall a, b \in \mathfrak{A}: (fa \sqsubseteq fb \Rightarrow fa \sqsubseteq f(a \sqcap b))$.
3. $\forall a, b \in \mathfrak{A}: (fa \sqsubseteq fb \Rightarrow fa \not\sqsupset f(a \sqcap b))$.
4. $\forall a, b \in \mathfrak{A}: (fa \sqsupset f(a \sqcap b) \Rightarrow fa \not\sqsubseteq fb)$.

Proof.

(1)⇔(2)⇔(3). Due $fa \sqsupseteq f(a \sqcap b)$.

(3)⇔(4). Obvious. □

Remark 3.3. The definition of straight map can be generalized for any poset $\mathfrak{A}$ by the formula

$$\forall a, b \in \mathfrak{A}: (fa \sqsubseteq fb \Rightarrow \exists c \in \mathfrak{A}: (c \sqsubseteq a \wedge c \sqsubseteq b \wedge fa = fc)).$$

This generalization is not yet researched however.

Proposition 3.4. Let f be a monotone map from a meet-semilattice $\mathfrak{A}$ to a meet-semilattice $\mathfrak{B}$. If

$$\forall a, b \in \mathfrak{A}: f(a \sqcap b) = fa \sqcap fb$$

then f is a straight map.

Proof. Let $fa \sqsubseteq fb$. Then $f(a \sqcap b) = fa \sqcap fb = fa$. □

Proposition 3.5. Let f be a monotone map from a meet-semilattice $\mathfrak{A}$ to some poset $\mathfrak{B}$. If

$$\forall a, b \in \mathfrak{A}: (fa \sqsubseteq fb \Rightarrow a \sqsubseteq b)$$

then f is a straight map.

Proof. $fa \sqsubseteq fb \Rightarrow a \sqsubseteq b \Rightarrow a = a \sqcap b \Rightarrow fa = f(a \sqcap b)$. □

Theorem 3.6. If f is a straight monotone map from a meet-semilattice $\mathfrak{A}$ then the following statements are equivalent:

1. f is an injection.